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关于调和数的等式*

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摘　要:对 Jonathon Peterson 的著名的二项式等式进行推广;用局部分解的方法获得一个关于调和数的新等式,应用等式可以获得一些另外的关于调和数的二项式等式.

关键词:二项式等式;局部分解;调和数

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1　预备知识

关于调和数定义:

$$H_n^{(r)}=\sum_{k=1}^{n}\frac{1}{k^r},H_n^{(r)}(-x)=\sum_{k=1,k\neq x}^{n}\frac{1}{(k-x)},H_n^{(r)}(x)=\sum_{k=0}^{n}\frac{1}{(k+x)^2}(x\neq-1,-2,\cdots)\tag{1}$$

引理 1　若 $m,n,r\in\mathbf{N},\theta\in\mathbf{R},r>n-m-2$,那么

$$\frac{(z+1)^2(z+2)^2\cdots(z+n)^2}{z^2(z-1)^2\cdots(z-m)^2(z-m+1)\cdots(z-n)}\frac{1}{(n-m)!}\frac{\theta^r}{(z+\theta)^r}=$$

$$\sum_{k=0}^{m}\binom{n}{k}\binom{n+k}{m+k}\binom{m}{k}\binom{m+k}{k}\binom{n+k}{k}\frac{(-1)^{n-m}\theta^r}{(k+\theta)^r}\left[\frac{1}{(z-k)^2}+\left(2H_{n+k}+H_{m-k}+H_{n-k}-4H_k-\frac{r}{\theta+k}\right)\frac{1}{z-k}\right]+$$

$$\sum_{k=m+1}^{n}\binom{n}{k}\binom{n+k}{m+k}\binom{m+k}{k}\binom{n+k}{k}\binom{k-1}{m}^{-1}\frac{(-1)^{n-k}\theta^r}{k(k+\theta)^r}\frac{1}{z-k}+\frac{\lambda}{(\theta+z)^r}+\cdots+\frac{\upsilon}{\theta+z}\tag{2}$$

证明　由局部分解可以得到

$$f(z)=\frac{(z+1)^2\cdots(z+n)^2}{z^2(z-1)^2\cdots(z-m)^2(z-m-1)\cdots(z-n)}\frac{1}{(n-m)!}\frac{\theta^r}{(z+\theta)^r}=$$

$$\sum_{k=0}^{m}\left(\frac{A_k}{(z-k)^2}+\frac{B_k}{z-k}\right)+\sum_{k=m+1}^{n}\frac{C_k}{z-k}+\frac{\lambda}{(z+\theta)^r}+\cdots+\frac{\upsilon}{z+\theta}$$

其中系数 A_k,B_k 和 C_k 待定.

当 $0\leqslant k\leqslant m$ 时,有

$$A_k=\lim_{z\to k}(z-k)^2f(z)=$$

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$$\lim_{z\to k}\frac{(z+1)^2\cdots(z+n)^2}{z^2\cdots(z-k+1)^2(z-k-1)^2(z-m)^2(z-m-1)\cdots(z-n)}\frac{1}{(n-m)!}\frac{\theta^r}{(\theta+z)^r}=$$

$$\frac{(k+1)^2\cdots(k+n)^2}{k^2\cdots1^2(-1)^2\cdots(k-m)^2(k-m-1)\cdots(k-n)}\frac{1}{(n-m)!}\frac{\theta^r}{(\theta+k)^r}=$$

$$\binom{n}{k}\binom{n+k}{m+k}\binom{n+k}{k}\binom{m}{k}\binom{m+k}{k}(-1)^{n-m}\frac{\theta^r}{(\theta+k)^r}$$

$$B_k=\lim_{z\to k}\frac{(z-k)^2f(z)-A_k}{z-k}=\lim_{z\to k}\frac{\mathrm{d}}{\mathrm{d}z}[(z-k)^2f(z)]=$$

$$\lim_{z\to k}\frac{\mathrm{d}}{\mathrm{d}z}\frac{(z+1)^2\cdots(z+n)^2}{z^2\cdots(z-k+1)^2(z-k-1)^2(z-m)^2(z-m-1)\cdots(z-n)}\frac{1}{(n-m)!}\frac{\theta^r}{(\theta+z)^r}=$$

$$\binom{n}{k}\binom{n+k}{m+k}\binom{n+k}{k}\binom{m}{k}\binom{m+k}{k}(-1)^{n-m}\frac{\theta^r}{(\theta+k)^r}\left(\sum_{i=1}^{n}\frac{2}{k+i}-\sum_{i=0,i\neq k}^{m}\frac{2}{k-i}-\sum_{i=m+1}^{n}\frac{1}{k-i}-\frac{r}{\theta+k}\right)=$$

$$\binom{n}{k}\binom{n+k}{m+k}\binom{n+k}{k}\binom{m}{k}\binom{m+k}{k}(-1)^{n-m}\frac{\theta^r}{(\theta+k)^r}\left(2H_{n+k}+H_{m-k}+H_{n-k}-4H_k-\frac{r}{\theta+k}\right)$$

当 $m+1\leqslant k\leqslant n$ 时,有

$$C_k=\lim_{z\to k}(z-k)f(z)=$$

$$\lim_{z\to k}\frac{(z+1)^2\cdots(z+n)^2}{z^2\cdots(z-m)^2(z-m-1)\cdots(z-k+1)(z-k-1)\cdots(z-n)}\frac{1}{(n-m)!}\frac{\theta^r}{(\theta+z)^r}=$$

$$\frac{(k+1)^2\cdots(k+n)^2}{k^2\cdots(k-m)^2(k-m-1)\cdots1(-1)\cdots(k-n)}\frac{1}{(n-m)!}\frac{\theta^r}{(\theta+k)^r}=$$

$$\binom{n}{k}\binom{n+k}{m+k}\binom{n+k}{k}\binom{m+k}{k}\binom{k-1}{m}^{-1}(-1)^{n-k}\frac{\theta^r}{(\theta+k)^r}$$

2 推广的新等式

Jonathon[2]给了一个简单的有趣的方法来证明二项式等式,即概率的方法,在这里将其推广并用局部分解及文献[3]中的方法证明,其中等式右边可以表示成贝尔多项式[4]的形式.

定理 1 若 $m,n,r\in\mathbf{N},\theta\in\mathbf{R},r>n-m-2,\theta>0$ 那么当 $\theta\notin\{1,2,\cdots,n\}$ 时,有

$$\sum_{k=0}^{m}\binom{n}{k}\binom{n+k}{m+k}\binom{m}{k}\binom{m+k}{k}\binom{n+k}{k}\frac{\theta^r}{(k+\theta)^r}\left(4H_k-H_{m-k}-H_{n-k}-2H_{n+k}+\frac{r}{\theta+k}\right)+$$

$$\sum_{k=m+1}^{n}\binom{n}{k}\binom{n+k}{m+k}\binom{m+k}{k}\binom{n+k}{k}\binom{k-1}{m}^{-1}\frac{(-1)^{m+k-1}\theta^r}{k(k+\theta)^r}=$$

$$\frac{1}{(n-m)!}\prod_{k=1}^{n}\left(\frac{k-\theta}{k+\theta}\right)^2\prod_{k=m+1}^{n}\frac{1}{\theta+k}\sum_{k_1+2k_2+\cdots=r-1}\frac{\theta^{r-2}}{k_1!\ k_2!\ \cdots}\left(\frac{W_1}{1}\right)^{k_1}\left(\frac{W_2}{2}\right)^{k_2}\cdots$$

当 $\theta\in\{1,2,\cdots,n\}$ 时,有

$$\sum_{k=0}^{m}\binom{n}{k}\binom{n+k}{m+k}\binom{m}{k}\binom{m+k}{k}\binom{n+k}{k}\frac{\theta^r}{(k+\theta)^r}\left(4H_k-H_{m-k}-H_{n-k}-2H_{n+k}+\frac{r}{\theta+k}\right)+$$

$$\sum_{k=m+1}^{n}\binom{n}{k}\binom{n+k}{m+k}\binom{m+k}{k}\binom{n+k}{k}\binom{k-1}{m}^{-1}\frac{(-1)^{m+k-1}\theta^{r}}{k(k+\theta)^{r}}=$$

$$\frac{1}{(n-m)!}\prod_{k=1,k\neq\theta}^{n}(k-\theta)^{2}\prod_{k=1}^{m}\frac{1}{\theta+k}\prod_{k=1}^{n}\frac{1}{\theta+k}\sum_{k_1+2k_2+\cdots=r-1}\frac{\theta^{r-2}}{k_1!\ k_2!\ \cdots}\left(\frac{W_1}{1}\right)^{k_1}\left(\frac{W_2}{2}\right)^{k_2}\cdots$$

其中 $W_k=(-1)^{k-1}2H_n^{(k)}(-\theta)+H_m^{(k)}(\theta)+H_n^{(k)}(\theta)$.

证明　将引理 1 中式(2)两边乘以 z,并使 $z\to\infty$ 可以得到

$$\sum_{k=0}^{m}\binom{n}{k}\binom{n+k}{m+k}\binom{m}{k}\binom{m+k}{k}\binom{n+k}{k}\frac{(-1)^{n-m}\theta^{r}}{(k+\theta)^{r}}\left(2H_{n+k}+H_{m-k}+H_{n-k}-4H_k-\frac{r}{\theta+k}\right)+$$
$$\sum_{k=m+1}^{n}\binom{n}{k}\binom{n+k}{m+k}\binom{m+k}{k}\binom{n+k}{k}\binom{k-1}{m}^{-1}\frac{(-1)^{n-k}\theta^{r}}{k(k+\theta)^{r}}+v=0 \tag{3}$$

当 $\theta\notin\{1,2,\cdots,n\}$ 时,有

$$v=[(\theta+z)^{-1}]\frac{(z+1)^{2}\cdots(z+n)^{2}}{z^{2}\cdots(z-m)^{2}(z-m-1)\cdots(z-n)}\frac{1}{(n-m)!}\frac{\theta^{r}}{(\theta+z)^{r}}=$$

$$[(\theta+z)^{r-1}]\frac{(z+1)^{2}\cdots(z+n)^{2}}{z^{2}\cdots(z-m)^{2}(z-m-1)\cdots(z-n)}\frac{\theta^{r}}{(n-m)!}=$$

$$[z^{r-1}]\frac{\left(1+\frac{z}{1-\theta}\right)^{2}\cdots\left(1+\frac{z}{n-\theta}\right)^{2}(-1)^{n-m}}{\left(1-\frac{z}{\theta}\right)^{2}\cdots\left(1-\frac{z}{\theta+m}\right)^{2}\left(1-\frac{z}{\theta+m+1}\right)\cdots\left(1-\frac{z}{\theta+n}\right)}\frac{\theta^{r-2}}{(n-m)!}\prod_{k=1}^{n}\left(\frac{k-\theta}{k+\theta}\right)^{2}\prod_{k=m+1}^{n}\frac{1}{\theta+k}=$$

$$\frac{(-1)^{n-m}\theta^{r-2}}{(n-m)!}\prod_{k=1}^{n}\left(\frac{k-\theta}{k+\theta}\right)^{2}\prod_{k=m+1}^{n}\frac{1}{\theta+k}[z^{r-1}]\exp\left(\sum_{k=1}^{\infty}\frac{z^{k}}{k}W_k\right)=$$

$$\frac{(-1)^{n-m}\theta^{r-2}}{(n-m)!}\prod_{k=1}^{n}\left(\frac{k-\theta}{k+\theta}\right)^{2}\prod_{k=m+1}^{n}\frac{1}{\theta+k}\sum_{k_1+2k_2+\cdots=r-1}\frac{1}{k_1!\ k_2!\ \cdots}\left(\frac{W_1}{1}\right)^{k_1}\left(\frac{W_2}{2}\right)^{k_2}\cdots$$

当 $\theta\in\{1,2,\cdots,n\}$ 时,有

$$v=[(\theta+z)^{-1}]\frac{(z+1)^{2}\cdots(z+n)^{2}}{z^{2}\cdots(z-m)^{2}(z-m-1)\cdots(z-n)}\frac{1}{(n-m)!}\frac{\theta^{r}}{(\theta+z)^{r}}=$$

$$[z^{r-3}]\frac{\left(1+\frac{z}{1-\theta}\right)^{2}\cdots\left(1+\frac{z}{\theta-1-\theta}\right)^{2}\left(1+\frac{z}{\theta+1-\theta}\right)^{2}\cdots\left(1+\frac{z}{n-\theta}\right)^{2}}{\left(1-\frac{z}{\theta}\right)^{2}\cdots\left(1-\frac{z}{\theta+m}\right)^{2}\left(1-\frac{z}{\theta+m+1}\right)\cdots\left(1-\frac{z}{\theta+n}\right)}$$

$$\frac{(-1)^{n-m}\theta^{r-2}}{(n-m)!}\prod_{k=1,k\neq\theta}^{n}(k-\theta)^{2}\prod_{k=1}^{n}\frac{1}{\theta+k}\prod_{k=1}^{m}\frac{1}{\theta+k}=$$

$$\frac{(-1)^{n-m}\theta^{r-2}}{(n-m)!}\prod_{k=1,k\neq\theta}^{n}(k-\theta)^{2}\prod_{k=1}^{n}\frac{1}{\theta+k}\prod_{k=1}^{m}\frac{1}{\theta+k}[z^{r-3}]\exp\left(\sum_{k=1}^{\infty}\frac{z^{k}}{k}W_k\right)=$$

$$\frac{(-1)^{n-m}\theta^{r-2}}{(n-m)!}\prod_{k=1,k\neq\theta}^{n}(k-\theta)^{2}\prod_{k=1}^{n}\frac{1}{\theta+k}\prod_{k=1}^{m}\frac{1}{\theta+k}\sum_{k_1+2k_2+\cdots=r-3}\frac{1}{k_1!\ k_2!\ \cdots}\left(\frac{W_1}{1}\right)^{k_1}\left(\frac{W_2}{2}\right)^{k_2}\cdots$$

3 应 用

推论1 设 $m=0,n,r\in\mathbf{N},\theta\in\mathbf{R},\theta>0$ 和 $r>n-2$,当 $\theta\notin\{1,2,\cdots,n\}$ 时,有

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}^{2}\frac{(-1)^{k-1}\theta^{r}}{k(k+\theta)^{r}}=3H_{n}-\frac{r}{\theta}+\frac{1}{n!}\prod_{k=1}^{n}\frac{(k-\theta)^{2}}{k+\theta}\sum_{k_1+2k_2+\cdots=r-1}\frac{\theta^{r-2}}{k_1!\ k_2!\ \cdots}\left(\frac{U_1}{1}\right)^{k_1}\left(\frac{U_2}{2}\right)^{k_2}\cdots$$

当 $\theta\in\{1,2,\cdots,n\}$ 时,有

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}^{2}\frac{(-1)^{k-1}\theta^{r}}{k(k+\theta)^{r}}=3H_{n}-\frac{r}{\theta}+\frac{1}{n!}\prod_{k=1,k\neq\theta}^{n}(k-\theta)^{2}\prod_{k=1}^{n}\frac{1}{k+\theta}\sum_{k_1+2k_2+\cdots=r-3}\frac{\theta^{r-2}}{k_1!\ k_2!\ \cdots}\left(\frac{U_1}{1}\right)^{k_1}\left(\frac{U_2}{2}\right)^{k_2}\cdots$$

其中 $U_k=(-1)^{k-1}2H_n^{(k)}(-\theta)+H_0^{(k)}(\theta)+H_n^{(k)}(\theta)$.

推论2 设 $m=n,n,r\in\mathbf{N},\theta\in\mathbf{R},\theta>0$ 和 $r\geqslant 0$,当 $\theta\notin\{1,2,\cdots,n\}$ 时,有

$$\sum_{k=0}^{n}\binom{n}{k}^{2}\binom{n+k}{k}^{2}\frac{\theta^{r}}{(k+\theta)^{r}}\left(4H_{k}-2H_{n-k}-2H_{n+k}+\frac{r}{\theta+k}\right)=$$

$$\prod_{k=1}^{n}\left(\frac{k-\theta}{k+\theta}\right)^{2}\sum_{k_1+2k_2+\cdots=r-1}\frac{\theta^{r-2}}{k_1!\ k_2!\ \cdots}\left(\frac{D_1}{1}\right)^{k_1}\left(\frac{D_2}{2}\right)^{k_2}\cdots$$

当 $\theta\in\{1,2,\cdots,n\}$ 时,有

$$\sum_{k=0}^{n}\binom{n}{k}^{2}\binom{n+k}{k}^{2}\frac{\theta^{r}}{(k+\theta)^{r}}\left(4H_{k}-2H_{n-k}-2H_{n+k}+\frac{r}{\theta+k}\right)=$$

$$\prod_{k=1,k\neq\theta}^{n}(k-\theta)^{2}\prod_{k=1}^{n}\left(\frac{1}{k+\theta}\right)^{2}\sum_{k_1+2k_2+\cdots=r-3}\frac{\theta^{r-2}}{k_1!\ k_2!\ \cdots}\left(\frac{D_1}{1}\right)^{k_1}\left(\frac{D_2}{2}\right)^{k_2}\cdots$$

其中 $D_k=(-1)^{k-1}2H_n^{(k)}(-\theta)+2H_n^{(k)}(\theta)$.

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An Equation Involving Harmonic Numbers

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Abstract: This paper mainly focuses on the extension of binomial identity by Jonathon. A new equation involving harmonic numbers is obtained by partial fraction decomposition, by which some other binomial identities involving harmonic numbers identities are obtained as well.

Key words: binomial identities; partial fraction decomposition; harmonic numbers