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a 尺度多小波正交尺度函数及其 mallat 算法

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摘 要:多小波解决了单小波不可能同时具有正交性、紧支性和对称性的困难,更具有研究的价值.在正交多小波理论的基础上,利用两尺度矩阵研究了一种特殊的紧支撑尺度函数构造成正交尺度函数的方法以及 a 尺度正交多小波 mallat 算法,得出了相应的分解和重构关系.

关键词:多小波;正交性;尺度函数;分解重构

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0 引 言

随着小波分析的发展日益成熟,其应用也越来越广泛,它的理论与算法也研究得越来越深入.由于 2 尺度小波的构造与 mallat 分解已成熟[1],如出现了由 I. Daubcchies 构造的一系列正交小波,而用 a 尺度小波分解信号可以得到更好的分辨率,灵活性更大,所以 a 尺度小波理论的研究也越来越重要,特别是正交多小波同时具有正交、对称、紧支等特点,研究 a 尺度正交多小波更具有重要意义[2-5]. 1996 年 Chui 和 Lian 利用对称性给出了 2 重多尺度函数和多小波,杨守志等人给出了几种多小波的构造方法.正交多尺度函数的构造方法有很多,此处基于一特殊的尺度函数,构造出正交多尺度函数,对正交尺度函数的构造有一定的价值. a 尺度多小波 Mallat 算法对信号的分解和去噪等分辨率更高,去噪效果更好,具有一定的研究意义.

1 a 尺度多分辨分析

定义 1[1] Hilbert 空间 $L^2(R)$ 中的一列闭子空间 $\{V_j\}_{j\in Z}$ 称为一个 a 尺度($a>1, a\in N_+$)正交多重多分辨分析,若满足:

1) $V_j \subseteq V_{j+1} (j\in Z)$.

2) $f(t)\in V_j \Leftrightarrow f(at)\in V_{j+1}$.

3) $\bigcap\limits_{j\in Z} V_j=\{0\}, \overline{\bigcup\limits_{j\in Z} V_j}=L^2(R)$.

4) 存在 r 个函数 $\varphi_1(t), \varphi_2(t), \cdots, \varphi_r(t)$,使 $\{\varphi_v(t-k), k\in Z, 1\leqslant v\leqslant r\}$ 为 V_0 的一个标准正交基,其中 $\varphi(t)=(\varphi_1(t), \varphi_2(t), \cdots, \varphi_r(t))^{\mathrm{T}}$ 为 a 尺度正交多分辨分析的尺度函数, $\varphi_{j,k}^v=a^{\frac{j}{2}}\varphi_v(a^j-k), \forall k, j\in Z$.

定义 $W_j, j\in Z$ 为 V_j 在 V_{j+1} 中的正交补,即 $V_{j+1}=V_j+W_j$,由小波分析理论可知存在 $\psi(t)=(\psi_1(t), \psi_2(t), \cdots, \psi_{(a-1)r}(t))^{\mathrm{T}}, \psi_i(t), i=1,2,\cdots,(a-1)r$,它们的伸缩与平移构成 W_0 的一个标准正交基,称 $\psi(t)$

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为正交多小波. 所以 $W_j=\mathrm{clos}_{L^2(R)}[\psi^{v}_{j,k}:1\leqslant v\leqslant(a-1)r,k\in Z],j\in Z$. 于是 $\varphi(t),\psi(t)$ 满足两尺度关系:

$$\begin{cases}\varphi(t)=\sum\limits_{k\in Z}P_k\varphi(at-k)\\ \psi(t)=\sum\limits_{k\in Z}Q_k\varphi(at-k)\end{cases}\tag{1}$$

其中 P_k 为两尺度矩阵序列,且 $P_k=(p_{u,kr+v})_{1\leqslant u,v\leqslant r},Q_k=(q_{u,kr+v})_{1\leqslant u\leqslant(a-1)r,1\leqslant v\leqslant r},k\in Z$ 分别为 $r\times r,(a-1)r\times r$ 阶矩阵,对式(1)作傅里叶变换得到 $\begin{cases}\hat\varphi(\omega)=P\left(\dfrac{\omega}{a}\right)\hat\varphi\left(\dfrac{\omega}{a}\right)\\ \hat\psi(\omega)=Q\left(\dfrac{\omega}{a}\right)\hat\psi\left(\dfrac{\omega}{a}\right)\end{cases}$,其中 $P(\omega)=\dfrac{1}{a}\sum\limits_{k\in Z}P_k\mathrm{e}^{-\mathrm{i}k\omega}$,$P(\omega)$ 为两尺度矩阵符号,$Q(\omega)=\dfrac{1}{a}\sum\limits_{k\in Z}Q_k\mathrm{e}^{-\mathrm{i}k\omega}$.

两个列向量值函数的内积定义为 $[A,B]=\int_R A\,\overline{B}^{\mathrm{T}}\mathrm{d}x$.

定义 2[2]　称 $\varphi(t)=(\varphi_1(t),\varphi_2(t),\cdots,\varphi_r(t))^{\mathrm{T}}$ 与 $\varphi(t)=(\varphi_1(t),\varphi_2(t),\cdots,\varphi_r(t))^{\mathrm{T}}$ 为 a 尺度 r 重正交多尺度函数,若满足 $[\varphi(t),\varphi(t-n)]=\delta_{0n}I_r,n\in Z$, $\psi(t)=(\psi_1(t),\psi_2(t),\cdots,\psi_{(a-1)r})^{\mathrm{T}}$ 与 $\psi(t)=(\psi_1(t),\psi_2(t),\cdots,\psi_{(a-1)r})^{\mathrm{T}}$ 为相应于 $\varphi(t)$ 的正交小波函数,若满足:

$$[\varphi(t),\psi(t-n)]=0,[\psi(t),\psi(t-n)]=\delta_{0,n}I_{(a-1)r},n\in Z$$

其中 0 为 $r\times(a-1)r$ 阶零矩阵,$I_{(a-1)t}$ 为 $(a-1)r$ 阶单位阵.

2　多小波正交尺度函数构造

引理 1[1,6,7]　设 $\varphi=(\varphi_1,\varphi_2,\cdots,\varphi_r)^{\mathrm{T}},\varphi_1,\varphi_2,\cdots,\varphi_r\in L(R)^2$,则 $\{\varphi_l(x-k):1\leqslant l\leqslant r,k\in Z\}$ 是一个正交族的充分必要条件是 $\sum\limits_{k\in Z}\hat\varphi(\omega+2k\pi)\overline{\hat\varphi^{\mathrm{T}}(\omega+2k\pi)}=I_r$.

引理 2　设 φ 是 a 尺度 r 重正交尺度函数,$P(z)$ 为两尺度符号 $\omega_j,j=1,2,\cdots,a$ 为 $z^a-1=0$ 的 a 个根,则 $\sum\limits_{j=1}^{a}P(\omega_jz)\overline{P^{\mathrm{T}}(\omega_jz)}=I_r,|z|=1$,等价于两尺度矩阵序列满足 $\sum\limits_{l\in Z}P_lP^{\mathrm{T}}_{l+ak}=a\delta_{0,k}I_r$.

证明　由引理 1 和两尺度方程可得

$$\begin{aligned}I_r=&\sum_{k\in Z}\hat\varphi(\omega+2k\pi)\overline{\hat\varphi^{\mathrm{T}}(\omega+2k\pi)}=\\&\sum_{k\in Z}P\left(\frac{\omega+2k\pi}{a}\right)\hat\varphi\left(\frac{\omega+2k\pi}{a}\right)\overline{\hat\varphi^{\mathrm{T}}\left(\frac{\omega+2k\pi}{a}\right)}\;\overline{P^{\mathrm{T}}\left(\frac{\omega+2k\pi}{a}\right)}=\\&\sum_{k=na}P\left(\frac{\omega}{a}+2n\pi\right)\hat\varphi\left(\frac{\omega}{a}+2n\pi\right)\overline{\hat\varphi^{\mathrm{T}}\left(\frac{\omega}{a}+2n\pi\right)}\;\overline{P^{\mathrm{T}}\left(\frac{\omega}{a}+2n\pi\right)}+\\&\sum_{k=na+1}P\left(\frac{\omega}{a}+\frac{\pi}{a}+2n\pi\right)\hat\varphi\left(\frac{\omega}{a}+\frac{\pi}{a}+2n\pi\right)\overline{\hat\varphi^{\mathrm{T}}\left(\frac{\omega}{a}+\frac{\pi}{a}+2n\pi\right)}\;\overline{P^{\mathrm{T}}\left(\frac{\omega}{a}+\frac{\pi}{a}+2n\pi\right)}+\cdots+\\&\sum_{K=na+a-1}P\left(\frac{\omega}{a}+\frac{(a-1)\pi}{a}+2n\pi\right)\hat\varphi\left(\frac{\omega}{a}+\frac{(a-1)\pi}{a}+2n\pi\right)\\&\overline{\hat\varphi^{\mathrm{T}}\left(\frac{\omega}{a}+\frac{(a-1)\pi}{a}+2n\pi\right)}\;\overline{P^{\mathrm{T}}\left(\frac{\omega}{a}+\frac{(a-1)\pi}{a}+2n\pi\right)}=\\&P(\omega_1z)\left[\sum_{n\in Z}\hat\varphi\left(\frac{\omega}{a}+2n\pi\right)\overline{\hat\varphi^{\mathrm{T}}\left(\frac{\omega}{a}+2n\pi\right)}\right]\overline{P^{\mathrm{T}}\left(\frac{\omega}{a}\right)}+\cdots+\\&P(\omega_az)\left[\sum_{n\in Z}\hat\varphi\left(\frac{\omega+(a-1)\pi}{a}+2n\pi\right)\overline{\hat\varphi^{\mathrm{T}}\left(\frac{\omega+(a-1)\pi}{a}+2n\pi\right)}\right]\overline{P^{\mathrm{T}}(\omega_az)}=\end{aligned}$$

$$\sum_{j=1}^{a} P(\omega_j z)\ \overline{P^{\mathrm{T}}(\omega_j z)}$$

展开即可得到 $\sum_{l\in Z} P_l P_{l+ak}^{\mathrm{T}} = a\delta_{0,k} I_r$.

命题 1 设 $P_k = \begin{cases} P_k, 0 \leqslant k < a \\ 0, \text{else} \end{cases}$ 是紧支尺度函数 φ 的两尺度序列,且 $\sum_{l\in Z} P_l P_l^{\mathrm{T}} \neq aI_r$,则存在着 $C, \tilde{P}_l = \sqrt{a}(C^{\mathrm{T}})^{-1} P_l$,使得 $\sum_{l\in Z} \tilde{P}_l \tilde{P}_{l+ak}^{\mathrm{T}} = aI_r$.

证明 由代数知识 $\sum_{l\in Z} P_l P_L^{\mathrm{T}}$ 是正定对称矩阵,所以存在着可逆矩阵 C,使得 $\sum_{l\in Z} P_l P_l^{\mathrm{T}} = C^{\mathrm{T}} C$,令 $\tilde{P}_l = \sqrt{a}(C^{\mathrm{T}})^{-1} P_l$,则 $\sum_{l\in Z} \tilde{P}_l \tilde{P}_l^{\mathrm{T}} = \sum_{l\in Z} a(C^{\mathrm{T}})^{-1} P_l P_l^{\mathrm{T}} C^{-1} = a(C^{\mathrm{T}})^{-1} \sum_{l\in Z} P_l P_l^{\mathrm{T}} C^{-1} = aI_r$, 又 $P_0 P_{ak} + P_1 P_{1+ak} + \cdots + P_{a-1} P_{a(k+1)-1} = 0$,由引理 2 即可得证.

由命题 1 及引理 2 就可以得到命题 2,构造一种较特殊的正交尺度函数.

命题 2 设 φ 为一尺度函数且两尺度序列满足 $P_k = \begin{cases} P_k, 0 \leqslant k < a \\ 0, \text{else} \end{cases}$, $\sum_{l} P_l P_{l+ak}^{\mathrm{T}} \neq aI_r$,则存在着 $\tilde{P}$,使得 $\tilde{\varphi}(t) = \sum_{l} \tilde{P}_l \varphi(at-l)$ 为一正交尺度函数. 这里 $\tilde{P}_l = \sqrt{a}(C^{\mathrm{T}})^{-1} P_l$.

例 1 为简便,不妨设 $a=2\varphi(t) = P_0\varphi(2t) + P_1\varphi(2t-1)$,这里 $P_0 = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1 \end{pmatrix}, P_1 = \begin{pmatrix} 1 & \\ & \sqrt{2} \end{pmatrix}$.

$$\sum_{l=0}^{1} P_l P_l^{\mathrm{T}} = P_0 P_0^{\mathrm{T}} + P_1 P_1^{\mathrm{T}} = \begin{pmatrix} 5 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix} = C^{\mathrm{T}} C$$

这里 $C^{\mathrm{T}} = \begin{pmatrix} \frac{\sqrt{26}+\sqrt{2}}{3} & \frac{2-\sqrt{13}}{3} \\ 1 & \sqrt{2} \end{pmatrix}, (C^{\mathrm{T}})^{-1} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{\sqrt{13}-2}{3} \\ \frac{-1}{\sqrt{13}} & \frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{3\sqrt{13}} \end{pmatrix}$,由命题 1 算得:

$$\tilde{P}_0 = \sqrt{2}(C^{\mathrm{T}})^{-1} P_0 = \begin{pmatrix} \frac{2\sqrt{2}}{\sqrt{13}} & \frac{\sqrt{2}}{3} + \frac{4\sqrt{2}}{3\sqrt{13}} \\ \frac{-2}{\sqrt{13}} & \frac{2}{3} - \frac{4}{3\sqrt{13}} \end{pmatrix}$$

$$\tilde{P}_1 = \sqrt{2}(C^{\mathrm{T}})^{-1} P_1 = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{2}{3} - \frac{4}{3\sqrt{13}} \\ \frac{-\sqrt{2}}{\sqrt{13}} & \frac{2\sqrt{2}}{3} + \frac{2\sqrt{2}}{3\sqrt{13}} \end{pmatrix}$$

则此时 $\tilde{\varphi}(t) = \tilde{P}_0\varphi(2t) + \tilde{P}_1\varphi(2t-1)$ 是一个正交尺度函数.

3 a 尺度正交多小波的 mallat 算法

定义投影算子:

$$P_j: L^2(R) \to V_j, P_j f = \sum_{k\in Z} \sum_{u=1}^{r} [f, \varphi_{jk}^u] \varphi_{jk}^u = \sum_{k\in Z} \sum_{u=1}^{r} c_{j,k}^u \varphi_{j,k}^u \tag{2}$$

$$Q_j: L^2(R) \to W_j, Q_j f = \sum_{k\in Z} \sum_{u=1}^{r} [f, \psi_{jk}^u] \psi_{jk}^u = \sum_{k\in Z} \sum_{u=1}^{(a-1)r} d_{j,k}^u \psi_{jk}^u \tag{3}$$

命题 3　设投影算子 P_j, Q_j 的定义如式(2)(3),则有下面的 mallat 快速算法.

1)分解算法:$\begin{cases} c^u_{j,l} = \dfrac{1}{\sqrt{a}} \sum\limits_{k\in Z}\sum\limits_{m=1}^{r} \overline{p_{u,(k-al)r+m}} c^m_{j+1,k} \\ d^u_{j,k} = \dfrac{1}{\sqrt{a}} \sum\limits_{k\in Z}\sum\limits_{m=1}^{r} \overline{q_{u,(k-al)r+m}} c^m_{j+1,k} \end{cases}$.

2)重构算法:$c^u_{j+1,l} = \dfrac{1}{\sqrt{a}} \sum\limits_{k\in Z}\sum\limits_{m=1}^{r} p_{m,(l-ak)r+u} c^m_{j,k} + \dfrac{1}{\sqrt{a}} \sum\limits_{k\in Z}\sum\limits_{m=1}^{(a-1)r} q_{m,(l-ak)r+u} d^m_{j,k}$.

证明　1) $\varphi^u_{j,l} = a^{\frac{j}{2}} \sum\limits_{k\in Z}\sum\limits_{m=1}^{r} p_{u,kr+m}\varphi_m(a(a^j t-l)-k) =$

$$\frac{1}{\sqrt{a}} \sum_{k\in Z}\sum_{m=1}^{r} p_{u,kr+m} a^{\frac{j+1}{2}} \varphi_m(a^{j+1}t - al - k) =$$

$$\frac{1}{\sqrt{a}} \sum_{k\in Z}\sum_{m=1}^{r} p_{u,(k-al)r+m} \varphi^m_{j+1,k}$$

$$c^u_{j,l} = [f, \varphi^u_{j,l}] = \left[f, \frac{1}{\sqrt{a}} \sum_{k\in Z}\sum_{m=1}^{r} p_{u,(k-al)r+m} \varphi^m_{j+1,k}\right] =$$

$$\frac{1}{\sqrt{a}} \sum_{k\in Z}\sum_{m=1}^{r} \overline{p_{u,(k-al)r+m}} [f, \varphi^m_{j+1,k}] =$$

$$\frac{1}{\sqrt{a}} \sum_{k\in Z}\sum_{m=1}^{r} \overline{p_{u,(k-al)r+m}} c^m_{j+1,k}$$

同理可证 $d^u_{j,l}$.

2) $c^u_{j+1,l} = [f, \varphi^u_{j+1,l}] = [P_j f + Q_j f, \varphi^u_{j+1,l}] =$

$$[P_j f, \varphi^u_{j+1,l}] + [Q_j f, \varphi^u_{j+1,l}] =$$

$$\left[\sum_{k\in Z}\sum_{m=1}^{r} c^m_{j,k}\varphi^m_{j,k}, \varphi^u_{j+1,l}\right] + \left[\sum_{k\in Z}\sum_{m=1}^{(a-1)r} d^m_{j,l}\psi^m_{j,k}, \varphi^u_{j=1,l}\right] =$$

$$\left[\sum_{k\in Z}\sum_{m=1}^{r} c^m_{j,k} \frac{1}{\sqrt{a}} \sum_{p\in Z}\sum_{\gamma=1}^{r} p_{m,(p-ak)r+\gamma}\varphi^u_{j+1,l}\right] + \left[\sum_{k\in Z}\sum_{m-1}^{(a-1)r} d^m_{j,k} \frac{1}{\sqrt{a}} \sum_{p\in Z}\sum_{\gamma=1}^{r} q_{m,(p-ak)r+\gamma}\varphi^{\gamma}_{j+1,p}, \varphi^u_{j+1,l}\right] =$$

$$\frac{1}{\sqrt{a}} \sum_{k\in Z}\sum_{m=1}^{r} p_{m,(l-ak)r+u} c^m_{j,k} + \frac{1}{\sqrt{a}} \sum_{k\in Z}\sum_{m=1}^{(a-1)r} q_{m,(l-ak)r+u} d^m_{j,k}$$

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Orthogonal Multiscaling Function and Mallat Algorithm with Dilation Factor a

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Abstract: Because multiwavelet solves the problem which single wavelet can not have orthogonality, compact support and symmetry simultaneously, as a result, multiwavelet has more value worth being studied. On the basis of orthogonal multiwavelet theory, by using two-scaling matrix, this paper studies orthogonal scaling function constructed by a special compact support scaling function and orthogonal multiwavelet Mallat algorithm with dilation factor a and obtains corresponding decomposition and reconstruction relation.

Key words: multiwavelet; orthogonality; scaling function; decomposition and reconstruction

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Analysis of New Functions of Input-Output Coefficients and Their Change

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Abstract: The direct consume coefficient is the most elementary coefficient for constructing input-output model. The direct distribution coefficient is the proportion of the quantity of production of all sections and the products of social final use provided by a department to the total quantity of the department. This paper explores again their new functions based on the past research and also makes an empirical analysis based on five input-output tables (1987-2007) of China.

Key words: direct consume coefficient; direct distribution coefficient; coefficient change

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