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Variance Stabilizing and Symmetrizing Transformations for Random Sum in Collective Risk Model Being Compound Negative Binomial Distributed*

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Abstract: By using Delta theorem, variance stabilizing transformation and symmetrizing transformation are studied for random sum in collective risk model being compound negative binomial distributed. These two transformations are both related to a sequence of independent, identically distributed random variables.

Key words: collective risk model; compound negative binomial distribution; variance stabilizing transformation; symmetrizing transformation; Delta theorem

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1 Introduction

Let N denote the number of claims arising from policies in a given time period. Let X_1 denote the amount of the first claim, X_2 the amount of the second claim and so on. In the collective risk model, the random sum $S_N = \sum_{i=1}^{N} X_i$ represents the aggregate claims generated by the portfolio for the period under study. The number of claims N is a random variable and is associated with the frequency of claim. The individual claims $X_1, X_2, \cdots$ are also random variables and are said to measure the severity of claims. There are two fundamental assumptions that we will make in this paper: $X_1, X_2, \cdots$ are identically distributed random variables with common distribution F and the random variables $N, X_1, X_2, \cdots$ are mutually independent.

When a negative binomial distribution is selected for N, the distribution of S_N is said to be a compound negative binomial distribution. The distribution of N is called a negative binomial distribution if $P(N=k) = \binom{r+k-1}{k} p^r q^k$, $k = 0, 1, 2, \cdots$, where $0 < p < 1$, $r \in N^+$ are parameters and $q = 1 - p$. In this case we write $N \sim Nb(r, p)$.

Through this paper, we will always assume that $N \sim Nb(r, p)$ and the random variables X_i, $i = 1, 2, \cdots$ have positive moments of all orders denoted by $\alpha_k = EX_1^k$, $k = 1, 2, \cdots$. Obviously,

$$ES_N = EN \cdot EX_1 = \frac{rq\alpha_1}{p}$$

$$\mathrm{Var}S_N = \mathrm{Var}[E(S_N|N)] + E[\mathrm{Var}(S_N|N)] = \mathrm{Var}(NEX_1) + E(N\mathrm{Var}X_1) =$$

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$$E^2X_1 \cdot \mathrm{Var}N + EN \cdot \mathrm{Var}X_1 = \frac{rq\alpha_2}{p} + r\left(\frac{q}{p}\right)^2\alpha_1^2$$

We assert that $Z_r = \frac{S_N - \frac{rq\alpha_1}{p}}{\sqrt{\frac{rq\alpha_2}{p} + r\left(\frac{q}{p}\right)^2\alpha_1^2}} \xrightarrow{L} N(0,1)$ as $r \to \infty$. We here give its proof.

Let $0 \equiv N_0 < N_1 < N_2 < \cdots N_r \equiv N$ such that $N_i - N_{i-1} \sim Nb(1,p)$, $i = 1,2,\cdots,r$. Then $\{S_{N_i}(N_{i-1}) = X_{N_{i-1}+1} + X_{N_{i-1}+2} + \cdots + X_{N_i}, i = 1,2,\cdots,r\}$, are sequence of independent and identical compound geometric distributed random variables, and $ES_{N_i}(N_{i-1}) = \frac{q\alpha_1}{p}$, and Var $S_{N_i}(N_{i-1}) = \frac{q}{p}\alpha_2 + \left(\frac{q}{p}\right)^2\alpha_1^2$.

from above, note that $S_N = \sum_{i=1}^{r} S_{N_i}(N_{i-1})$ and Lévy central limit theorem finishes our assertion. Anscombe [1] has showed the variance stabilizing transformation of the negative binomial distribution is $\sqrt{k - \frac{1}{2}}\sinh^{-1}\sqrt{\frac{r + \frac{3}{8}}{k - \frac{3}{4}}}$, and DasGupta [2] gave the variance stabilizing transformation and the symmetrizing transformation of the Poisson, binomial distribution. All of them only consider the transformation of the single distribution. In this paper, we take the compound negative binomial distribution into account. The form of the transformation of the compound negative binomial distribution is completely different from the form of the negative binomial case. In section 2, we consider a variance stabilizing transformation of S_N in the form $\sqrt{S_N}$. In section 3, by using the method introduced by [2], we get the symmetrizing transformation of the form $S_N^{1-c_1}$, $c_1 > 0$.

2 Variance stabilizing transformation of S_N

From above, we know that $\sqrt{r}\left(\frac{S_N}{r} - \frac{q}{p}\alpha_1\right) \xrightarrow{L} N\left(0, \frac{q\alpha_2}{p} + \left(\frac{q}{p}\right)^2\alpha_1^2\right)$ as $r \to \infty$. Make some changes, we can get that

$$\frac{S_N - r\frac{q}{p}\alpha_1}{\sqrt{r\frac{q}{p}\alpha_2 + r\left(\frac{q}{p}\right)^2\alpha_1^2}} \xrightarrow{L} N(0,1)$$

as $r \to \infty$.

Let $\gamma = r\frac{q}{p}\alpha_1$, $\delta(\gamma) = \sqrt{\gamma\frac{\alpha_2}{\alpha_1} + \gamma\frac{q}{p}\alpha_1}$, by Delta theorem, a variance stabilizing transformation of S_N is $g(\gamma) = \int \frac{k}{\delta(\gamma)}d\gamma = \int \frac{k}{\sqrt{\gamma c}}dr = 2k\sqrt{\frac{\gamma}{c}}$, where $c = \alpha_2 + \frac{q}{p}\alpha_1$ will be a constant if p and the distribution of X_i are given. Taking $k = \frac{\sqrt{c}}{2}$ to give that $g(\gamma) = \sqrt{\gamma}$ is a variance stabilizing transformation for the compound negative binomial distribution. We can obtain that $\sqrt{S_N} - \sqrt{r\frac{q}{p}\alpha_1} \xrightarrow{L} N\left(0, \frac{c}{4}\right)$.

Consider the Taylor series expansions of $\sqrt{S_N}$ about the point $\frac{rq\alpha_1}{p}$, then we have:

$$\sqrt{S_N}=\sqrt{\frac{rq\alpha_1}{p}}\left\{\begin{array}{l}1+a_1\dfrac{S_N-\frac{rq\alpha_1}{p}}{\frac{rq\alpha_1}{p}}-a_2\left(\dfrac{S_N-\frac{rq\alpha_1}{p}}{\frac{rq\alpha_1}{p}}\right)^2+\cdots\\+(-1)^s a_{s-1}\left(\dfrac{S_N-\frac{rq\alpha_1}{p}}{\frac{rq\alpha_1}{p}}\right)^{s-1}\end{array}\right\}+R_s \tag{1}$$

where $a_s=(-1)^{s+1}\frac{1\cdot(-1)\cdot(-3)\cdots(-2s+3)}{2^s\cdot s!}$, $s=1,2,\cdots$, $R_s=\sum_{i=s}^{\infty}(-1)^{i+1}a_i\frac{(S_N-r\frac{q}{p}\alpha_1)^i}{(r\frac{q}{p}\alpha_1)^{i-\frac{1}{2}}}$. Taking expectations of both sides of (1), we obtain:

$$E(\sqrt{S_N})=\sqrt{r\frac{q}{p}\alpha_1}-\frac{1}{4}\left(\frac{q}{p}\alpha_2+\left(\frac{q}{p}\right)^2\alpha_1^2\right)r^{-\frac{1}{2}}\left(\frac{q}{p}\alpha_1\right)^{-\frac{3}{2}}+O(r^{-\frac{3}{2}})$$

$$\begin{aligned}\operatorname{Var}(\sqrt{S_N})=&\frac{\alpha_2+(\frac{q}{p})\alpha_1^2}{4\alpha_1}+\frac{1}{16r}\left\{\frac{3q}{p}\alpha_1+\frac{6\alpha_2}{\alpha_1}+\frac{18\alpha_2}{\alpha_1^2}+\frac{3\alpha_2^2+3\alpha_2\alpha_3+\alpha_3}{\frac{q}{p}\alpha_1^3}-\right.\\&\left.4\frac{\alpha_2}{\frac{q}{p}\alpha_1}-4\frac{\alpha_3+3\frac{q}{p}\alpha_1\alpha_2+2(\frac{q}{p})^2\alpha_1^3}{\frac{q}{p}\alpha_1^2}+\frac{\left(\alpha_2+\frac{q\alpha_1^2}{p}\right)^2}{(\frac{q}{p})(\alpha_1)^3}\right\}+\\&\frac{1}{r^2}\left\{6\frac{q}{p}\alpha_1+6\frac{\alpha_2}{\alpha_1}+6\frac{\alpha_2}{\alpha_1^2}+\frac{3\alpha_2^2+3\alpha_1\alpha_2+\alpha_3}{\frac{q}{p}\alpha_1^3}+\frac{\alpha_4}{(\frac{q}{p})^2\alpha_1^3}-\frac{1}{8}\frac{\frac{q}{p}}{\alpha_1^3}\left(\alpha_2+\frac{q}{p}\alpha_1^2\right)^2\right\}+O(r^{-3})\end{aligned}$$

We conclude that the transformation has a variance – bias.

3 Symmetrizing transformation of S_N

We don't directly find the symmetrizing transformation of S_N. The method introduced by DasGupta[2] can well solve this problem directly.

Suppose S_{N_j}, $j=1,2,\cdots,n$ are independent and identically distributed random variables with common compound negative binomial distribution as above, then $\frac{\sqrt{n}(\overline{S_N}-r\frac{q}{p}\alpha_1)}{\sqrt{r\frac{q}{p}\alpha_2+r(\frac{q}{p})^2\alpha_1^2}}\xrightarrow{L}N(0,1)$, $E(\overline{S_N}-r\frac{q}{p}\alpha_1)=0$,

$\operatorname{Var}(\overline{S_N})=\frac{r\frac{q}{p}\alpha_2+r(\frac{q}{p})^2\alpha_1^2}{n}$, $E(\overline{S_N}-r\frac{q}{p}\alpha_1)^3=\frac{\left[r\frac{q\alpha_3}{p}+3r(\frac{q}{p})^2\alpha_1\alpha_2+2r(\frac{q\alpha_1}{p})^3\right]}{n^2}$.

Let $\beta=r\frac{q}{p}\alpha_1>0$. From above, we have that $b(\beta)=0$, $\sigma(\beta)=(\beta\frac{\alpha_2}{\alpha_1}+\beta\frac{q}{p}\alpha_1)^{\frac{1}{2}}$ and

$$d(\beta)=\frac{d_{31}(\beta)}{\sigma^3(\beta)}=\frac{\beta\frac{\alpha_3}{\alpha_1}+3\beta\frac{q}{p}\alpha_2+2\beta(\frac{q}{p})^2\alpha_1^2}{(\beta\frac{\alpha_2}{\alpha_1}+\beta\frac{q}{p}\alpha_1)^{\frac{3}{2}}}=\frac{\frac{\alpha_3}{\alpha_1}+3\frac{q}{p}\alpha_2+2(\frac{q}{p})^2\alpha_1^2}{\beta^{\frac{1}{2}}(\frac{\alpha_2}{\alpha_1}+\frac{q}{p}\alpha_1)^{\frac{3}{2}}}$$

Then the symmetrizing transformation of $\overline{S_N}$ is the solution of the differential equation: $d(\beta) + 3\sigma(\beta)\frac{g''(\beta)}{g'(\beta)} = 0$, which is equivalent that:

$$\frac{\frac{\alpha_3}{\alpha_1} + 3\frac{q}{p}\alpha_2 + 2\left(\frac{q}{p}\right)^2\alpha_1^2}{\beta^{\frac{1}{2}}\left(\frac{\alpha_2}{\alpha_1} + \frac{q}{p}\alpha_1\right)^{\frac{3}{2}}} + 3\left(\beta\frac{\alpha_2}{\alpha_1} + \beta\frac{q}{p}\alpha_1\right)^{\frac{1}{2}}\frac{g''(\beta)}{g'(\beta)} = 0 \tag{2}$$

The solution of (2) is $g(\beta) = \frac{c_2}{1-c_1}\beta^{1-c_1}$, where $c_1 = \frac{\frac{\alpha_3}{\alpha_1} + 3\frac{q\alpha_2}{p} + 2\left(\frac{q\alpha_1}{p}\right)^2}{3\left(\frac{\alpha_2}{\alpha_1} + \frac{q\alpha_1}{p}\right)^2}$ and c_2 is constant. $g(\beta)$ is decided by c_1, and providing that p is given, c_1 will change with the different distribution of X_j, so the symmetrizing transformation will be decided by the distribution of X_j.

From above, we can easily obtain that $g(S_N) = \frac{c_2}{1-c_1}S_N^{1-c_1}$ is the symmetrizing transformation of S_N. $g(S_N) = S_N^{1-c_1}$ is also the symmetrizing transformation of S_N. Note that for $p = q = \frac{1}{2}$, $\alpha_i = 1$, $c_1 = \frac{1}{2}$, $\sqrt{S_N}$ is a symmetrizing transformation.

References:

[1] ANSCOMBE F J. The transformation of Poisson, binomial, and negative binomial data [J]. Biometrika, 1948, 35: 246-254

[2] DAS G A. Asymptotic Theory of Statistics and Probability [M]. Springer, New York, 2008

聚合风险模型中的随机和服从复合负二项分布时的方差稳定变换和对称变换

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摘 要:利用 Delta 定理研究聚合风险模型中的随机和服从复合负二项分布时的方差稳定变换和对称变换,通过研究发现方差稳定变换和对称变换都涉及独立同分布随机变量.

关键词:聚合风险模型; 复合负二项分布; 方差稳定变换; 对称变换; Delta 定理

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