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The optimality conditions of infinite vector optimization problems

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Abstract: In paper [1], the author established the optimality conditions for the finite dimensional multiobjective differentiable programming, and got a series of meaningful conclusions. In this paper, we expand these conclusions to the infinite vector optimation problems, and the optimality conditions of infinite vector optimation problems are obtained.

Key words: infinite dimensional spaces; constraint qualification; necessary optimality condition; sufficient optimality condition; Pareto-optimal solutions; η - pseudoconvexity

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Many results of the optimality conditions for infinite vector optimation problems have already caused the interest of a lot of scholars. This paper deals with the infinite vector optimation problems. By using the constraint qualification $S \subseteq C(K, x^0)$, the necessary optimality conditions for weak Pareto-optimal solutions are obtained. The sufficient optimality conditions for weak Pareto-optimal solutions are discussed in the case of convexity and η - convexity separately.

1 Preliminaries

Let X be a Banach space and Y be a locally convex Hausdorff space. Suppose that Y_+ is a positive cone with nonempty interior in Y, $Y_+ \neq Y$, $0 \in Y_+$. We may obtain a linear order $\leq$ on Y defined by $y' \leq y$ if $y - y' \in Y_+$.

But, $y' < y$ means $y - y' \in \mathrm{int} Y_+$. $y \leq y' \Leftrightarrow y' \geq y$, $y < y' \Leftrightarrow y' > y$. Let Y^* denote the topological dual of Y. The dual cone Y_+^* of Y_+ is denoted by $Y_+^* = \{ y^* \in Y^* : \langle y, y^* \rangle \geq 0, \forall y \in Y_+ \}$, where $\langle y, y^* \rangle$ denotes the value of the continuous linear functional y^* at the point y. In this paper, consider the infinite vector optimation problems:

$$\begin{aligned} &\min f(x) \\ &\text{s.t.} \ -g(x) \in Z_+, \ h(x) = 0 \end{aligned} \qquad \text{(VP)}$$

where $f: X \to Y$, $g: X \to Z$, and $h: X \to W$ are all differentiable functions in the sense of Frechet (or simply F-differentiable). Let Y have a positive cone with nonempty topological interior. let Z, W be Banach spaces with positive cones Z_+ and W_+ with nonempty topological interiors respectively, and W_+ be a point cone.

Let $K = \{ x \in X: g(x) \leq 0, h(x) = 0 \}$, where K is a feasible set of the problem (VP).

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Definition 1 We say that $x^0 \in K$ is a Pareto-optimal solution of the problem (VP), if there exists no $x \in K$, $x \neq x^0$ such that $f(x) \leq f(x^0)$.

Definition 2 We say that $x^0 \in K$ is a weak Pareto-optimal solution of the problem (VP), if there does not exist $x \in K$ such that $f(x) \leq f(x^0)$.

Lemma 1 (alternative theorem, see [2]) Let D be a nonempty set and Y be an ordered linear topological space with a positive cone Y_+ with nonempty interior. If $F: D \to Y$ is a subconvexlike mapping, then either (i) or (ii) holds:

(**i**) there is $x^0 \in D$ such that $F(x^0) < 0$;

(**ii**) there is $y^* \in Y_+^*$ such that $y^* \neq 0$ and $\langle F(x), y^* \rangle \geq 0$, $\forall x \in D$.

The two alternatives (**i**) and (**ii**) exclude each other.

2 Necessary optimality conditions

Definition 3 Let V be a locally convex Hausdorff space, $U \subseteq V$. The vector $d \in V$ is called a convergence vector for U at $u_0 \in U$, if and only if there exists a sequence $\{u_n\}$ in U and a sequence $\{a_n\}$ of a positive real numbers such that:

$$\lim_{n\to\infty} u_n = u_0, \lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} \frac{u_n - u_0}{a_n} = d$$

Lemma 2 (see [6]) Suppose that $x^0 \in K$ is a weak Pareto-optimal solution of the problem (VP), then no convergence vector for $f(K)$ at $f(x^0)$ is strictly negative.

Let $C(K, x^0)$ be the set of all convergence vectors for K at x^0, and let $S = \{x \in X: g'(x^0)x \leq 0, h'(x^0)x = 0\}$.

We say that g and h satisfy a constraint qualification at x^0 if $S \subseteq C(K, x^0)$.

Theorem 1 Suppose that:

(i) $x^0 \in K$ is a weak Pareto-optimal solution of (VP);

(ii) f, g and h are F - differentiable at x^0;

(iii) g and h satisfy the constraint qualification at x^0.

then, there exist $y_0^* \in Y_+^*$, $y_0^* \neq 0$, $z_0^* \in Z_+^*$, $w_0^* \in W^*$, such that:

$$y_0^* f'(x^0) + z_0^* g'(x^0) + w_0^* h'(x^0) = 0$$
$$g(x^0) \leq 0, \ h(x^0) = 0$$

Proof Let x^0 be a weak Pareto-optimal solution of (VP). Then $x^0 \in K$ and hence $g(x^0) \leq 0$, $h(x^0) = 0$.

(i) Let $d \in C(K, x^0)$. Then, there exists a sequence $\{x_n\} \subset K$ and a sequence $\{a_n\}$ of a positive real numbers such that:

$$x_n \to x^0, a_n \to 0, \frac{x_n - x^0}{a_n} \to d$$

Due to f is F - differentiable at x^0, we have:

$$f(x_n) = f(x^0) + f'(x^0)(x_n - x^0) + \mathbf{o}(x_n - x^0), \text{ i.e.}, \frac{f(x_n) - f(x^0)}{a_n} = f'(x^0)\frac{x_n - x^0}{a_n} + \frac{\mathbf{o}(x_n - x^0)}{x_n - x^0} \cdot \frac{x_n - x^0}{a_n}$$

where $o(x_n - x^0) / \| x_n - x^0 \| \to 0 \ (n \to \infty)$. Therefore, $\frac{f(x_n) - f(x^0)}{a_n} \to f'(x^0)d$.

By the continuity of f, $f(x_n) - f(x^0)$. It means that $f'(x^0)d$ is a convergence vector of $f(K)$ at $f(x^0)$. By lemma 2, $f'(x^0)d \not< 0$, Note that W have a positive cone W_+ with nonempty topological interior, and W_+ is a point cone. Hence, for any $d \in C(K, x^0)$, the system of $f'(x^0)d < 0$, $g'(x^0)d \leq 0$, $-h'(x^0)d \leq 0$, $h'(x^0)d \leq 0$ is inconsistent

(**ii**) Let $d \in X$, $d \notin C(K, x^0)$, then $d \notin S$. That is, for any $d \notin C(K, x^0)$, the system $g'(x^0)d \leq 0$, $-h'(x^0)$

$d\leq 0, h'(x^0)d\leq 0$ is inconsistent. Hence, for any $d\notin C(K, x^0)$, the system $f'(x^0)d<0, g'(x^0)d\leq 0, -h'(x^0)d\leq 0, h'(x^0)d\leq 0$ is inconsistent.

Let $F(x)=(f'(x^0)x, g'(x^0)x, -h'(x^0)x, h'(x^0)x)$, $x\in X$.

We note that Y_+, Z_+, W_+ have nonempty topological interiors. Hence, there exists no $d\in X$, such that $F(d)=(f'(x^0)d, g'(x^0)d, -h'(x^0)d, h'(x^0)d)<0$.

It is obvious that X is a convex set. Since F is a convex function on X, it is a subconvexlike mapping. According to Lemma 1, $\exists p^*=(y_0^*, z_0^*, u_0^*, v_0^*)\in Y_+^*\times Z_+^*\times W_+^*\times W_+^*$, $p^*\neq 0$, such that $[F(d), p^*]\geq 0, \forall d\in X$. i.e., $y_0^* f'(x^0)d+z^* g'(x^0)d-u_0^* h'(x^0)d+v_0^* h'(x^0)d\geq 0$.

While changing into the $-d$, $y_0^* f'(x^0)d+z^* g'(x^0)d-u_0^* h'(x^0)d+v_0^* h'(x^0)d\leq 0$.

Hence, we get $y_0^* f'(x^0)d+z^* g'(x^0)d-u_0^* h'(x^0)d+v_0^* h'(x^0)d=0$.

Let $w_0^*=v_0^*-u_0^*\in W^*$. We get:

$$y_0^* f'(x^0)d+z^* g'(x^0)d+w_0^* h'(x^0)d=0, \qquad \forall d\in X \tag{1}$$

From (1), we obtain:

$$y_0^* f'(x^0)+z_0^* g'(x^0)+w_0^* h'(x^0)=0$$

3 Sufficient optimality conditions

Corollary 1 The conclusions of Theorem 1 hold with hypothesis (i) and (ii) as above, but hypothesis (iii) replaced by (iii)′ $S_h=\{x\in X: h'(x^0)x=0\}\subseteq C(K, x^0)$.

Since $S\subseteq S_h$, the corollary establishs obviously.

Theorem 2 Suppose that:

(i) f, g and h are F-differentiable at x^0;

(ii) f and g are convex, h is convex and concave;

(iii) there exist $y_0^*\in Y_+^*, y_0^*\neq 0, z_0^*\in Z_+^*, w_0^*\in W^*$, such that:

(a) $y_0^* f'(x^0)x+z_0^* g'(x^0)x+w_0^* h'(x^0)x\geq 0, \forall x\in S$;

(b) $g(x^0)=0$;

(c) $h(x^0)=0$.

Then, x^0 is a weak Pareto-optimal solution of (VP).

Proof If x^0 is not a weak Pareto-optimal solution of (VP), then, there exists $x^*\in K$ such that $f(x^*)<f(x^0)$. Hence, by assumptions (i), (ii), (iii)(b), (iii)(c), we have:

$$f'(x^0)(x^*-x^0)\leq f(x^*)-f(x^0)<0$$
$$g'(x^0)(x^*-x^0)\leq g(x^*)-g(x^0)\leq 0$$
$$h'(x^0)(x^*-x^0)=h(x^*)-h(x^0)=0$$

It follows that $\exists x^*-x^0\in S$, such that $f'(x^0)(x^*-x^0)<0$. For $f'(x^0)$ using on S in Lemma 1, by this time the assumption (ii) of Lemma 1 does not hold for $y_0^*\in Y_+^*, y_0^*\neq 0$. Therefore, we have $x'\in S$ such that:

$$y_0^* f'(x^0)x'<0 \tag{2}$$

Using $x'\in S$, we obtain $g'(x^0)x'\leq 0, h'(x^0)x'=0$. Since $z_0^*\in Z_+^*$ and $w_0^*\in W^*$, therefore:

$$z_0^* g'(x^0)x'\leq 0, w_0^* h'(x^0)x'=0 \tag{3}$$

By (2) and (3), we get $y_0^* f'(x^0)x'+z_0^* g'(x^0)x'+w_0^* h'(x^0)x'<0$, which contradicts (iii)(a). Consequently, x^0 is a weak Pareto-optimal solution of (VP).

In the next theorem, we replace the convexity of f, g and h by the η-pseudoconvexity of a suitable linear combination of the components of f, g and h.

Defitination 4 Let $\varphi: X\to Y$ is F-differentiable at x^0 on X. The mapping φ is said to be η-convexity at x^0, if

$\exists\eta: X \times X \to X$ such that:

$$\phi(x) - \phi(x^0) \geq \phi'(x^0)\eta(x, x^0), \forall x \in X$$

The mapping ϕ is said to be η - pseudoconvexity at x^0, if $\phi'(x^0)\eta(x, x^0) \geq 0 \Rightarrow \phi(x) \geq \phi(x^0), \forall x \in X$.

Theorem 3 Suppose that:

(i) f, g and h are F - differentiable at x^0;

(ii) there exist $y_0^* \in Y_+^*$, $y_0^* \neq 0$, $z_0^* \in Z_+^*$, $w_0^* \in W^*$, such that:

(a) $[y_0^* f'(x^0) + z_0^* g'(x^0) + w_0^* h'(x^0)]\eta(x, x^0) \geq 0, \forall x \in K$;

(b) $g(x^0) = 0$;

(c) $h(x^0) = 0$.

(iii) $y_0^* f(x) + z_0^* g(x) + w_0^* h(x)$ is η - pseudoconvexity at x^0.

Then, x^0 is a weak Pareto-optimal solution of (VP).

Proof If x^0 is not a weak Pareto-optimal solution of (VP), there exists $x^* \in K$ such that $f(x^*) - f(x^0) < 0$. Since $x^* \in K$, by assumptions (ii) (b) and (ii) (c), therefore, we conclnde that:

$$g(x^*) - g(x^0) \leq 0, \ h(x^*) - h(x^0) = 0$$

$$y_0^* f(x^*) + z_0^* g(x^*) + w_0^* h(x^*) < y_0^* f(x^0) + z_0^* g(x^0) + w_0^* h(x^0) \tag{4}$$

If $[y_0^* f(x^0) + z_0^* g(x^0) + w_0^* h(x^0)]'_{x=x^0}\eta(x^*, x^0) = [y_0^* of'(x^0) + z_0^* og'(x^0) + w_0^* oh'(x^0)]\eta(x^*, x^0) \geq 0$.

By definition of η - pseudoconvexity, we have:

$$y_0^* f(x^*) + z_0^* g(x^*) + w_0^* h(x^*) \geq y_0^* f(x^0) + z_0^* g(x^0) + w_0^* h(x^0)$$

which contradicts (4). Therefore, we obtain:

$$[y_0^* f'(x^0) + z_0^* g'(x^0) + w_0^* h'(x^0)]\eta(x, x^0) < 0, \forall x \in K$$

but this violates hypothesis (ii) (a). Hence x^0 is a weak Pareto-optimal solution of (VP).

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无穷维向量最优化问题的最优性条件

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摘 要:文[1]在有穷维空间中建立了可微多目标规划的最优性条件,并得出了一些有意义的结论. 此处将这些结论推广到了无穷维空间中,得到了无穷维空间中向量最优化问题的最优性条件.

关键词:无穷维空间;约束规格;最优性必要条件;最优性充分条件; Pareto最优解;η—伪凸

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