文章编号: 1672 - 058X(2009) 03 - 0213 - 06

# Numerical simulation for the infiltration problems of water flow*

LI Huan-rong
(College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China)

**Abstract:** In this paper, a numerical model for an unsaturated soil water flow equation is established by the finite volume element(FVE) methods The numerical method has been verified and compared by the numerical examples Satisfactory results and some other significant and valuable conclusions are obtained

**Key words:** Numerical simulation; Water flow problems; Infiltration Problems
**CIC number:** O337. 3 **Document code:** A

The flow of unsaturated soil water, which is a flow as the soil holes are not full of water, is an important form of flow in porous media The prediction for the unsaturated flow is of significance in many branches of science and engineering including atmospheric science, soil science, agricultural engineering, environmental engineering and groundwater hydrology, and so on Soil water density is a crucial climate factor, and its seasonal change casts important influence on weather and climate in mid-high latitude region Land surface parameterization which stresses on computation of soil moisture density has been widely concerned[1,2]. All hydraulic processes at surface and subsurface of the earth, such as precipitation, evaporation and evapotranspiration, seepage of surface water, capillary elevation of deep-level water, absorption in root zone and liquid moisture flow of groundwater, and so forth, can all be reduced to unsaturated flow problem. Since the problem is described by a nonlinear equation, it is impossible to obtain its analytical solution except for special cases Therefore, numerical approximations[3,4] are typically used to solve the unsaturated flow equation The finite element methods and the finite dierence methods for the unsaturated flow equation have been studied by several authors[5,6]. However, to the finite dierence methods, the changes of boundary condition and soil parameters easily cast apparent influence on error estimates, and the finite element methods require fairly more computational expenses, so it is not convenient to apply them to numerical simulation in the actual land surface model

In the paper, a numerical model for an unsaturated soil water flow equation is established by the finite volume element methods[7], and some numerical examples is given

## 1 Semi-discrete finite volume element schemes

Based on the horizontal resolution of the general atmosphere circulation model (usually 1 ° to 5 ° longitude-lati-

收稿日期: 2009 - 01 - 15;修回日期: 2009 - 03 - 10.
*基金项目:重庆市教委科技研究项目 (KJ080716) ,重庆工商大学青年教师基金科研项目 (0752013)和学生科技创新项目 (083018)资助.
作者简介:李焕荣 (1979 - ) ,女 ,山东泰安人 ,副教授 ,博士 ,从事偏微分方程数值解、土壤水动力学研究.

tudes resolution), if the liquid moisture flow in soil along horizontal direction may be ignored, it can be reduced to a one-dimensional unsaturated soil flow (problem (1)). Let $z$ denote the vertical dimension, assuming positive downward, and $Q(z, t)$ be soil volumetric moisture density at time $t$ and distance $z$ from the surface. Suppose that infiltration or evaporation rate at the surface is dependent on time and to be given, positive is for infiltration and negative is for evaporation. Let $I = (0, L)$, and moisture density at the bottom of the domain $I$ be given, independent of distance. Then, by Darcy law and the continuous principle, the Richards equation of unsaturated flow can be described as (see [3-6] for detail):

$$\begin{cases} \dfrac{\partial Q}{\partial t} - \dfrac{\partial}{\partial z}\left( D(Q) \dfrac{\partial Q}{\partial z} \right) + \dfrac{\partial K(Q)}{\partial z} = S_r, & z \in I,\ t \in (0, T) \\ Q(z, 0) = Q_0(z), & z \in I \\ \dfrac{\partial Q}{\partial z}(L, t) = 0, & t \in (0, T) \\ Q(0, t) = Q_1(t), & t \in (0, T) \end{cases} \tag{1}$$

where $Q$ is the soil moisture density, $-S_r$ the absorption rate of root zone, $K(Q)$ the unsaturated hydraulic conductivity, $D(Q)$ the soil water diffusivity, $Q_1(z)$ the given moisture density at upper boundary $z = 0$, and $Q_0(z)$ the given initial condition. The relationships between the hydraulic conductivity $K(Q)$, the soil water diffusivity $D(Q)$ and $Q$ are as follows:

$$\begin{cases} K(Q) = K_s \left( \dfrac{Q}{Q_s} \right)^{2b+3} \\ D(Q) = - \dfrac{b K_s \psi_s}{Q_s} \left( \dfrac{Q}{Q_s} \right)^{b+2}, Q_r \le Q(Zz, t) \le Q_s \end{cases} \tag{2}$$

where $Q_r$ is the residual moisture density, $Q_s$ the saturated moisture density and $0 < Q_s < 1$, $K_s$ the saturated hydraulic conductivity, $b$ the soil parameter and $\psi_s$ defining change in soil water, all dependent on soil. Obviously, $K(Q)$, $\dfrac{\partial K(Q)}{\partial z}$, $D(Q)$ and $\dfrac{\partial D(Q)}{\partial Q}$ are bounded from above and below, i. e., there exist two constants $K_1$ and $K_2$, such that:

$$K_1 \le K(Q) \le, \frac{\partial K(Q)}{\partial Q}, \frac{\partial K(Q)}{\partial z}, D(Q), \frac{\partial D(Q)}{\partial Q} \le K_2 \tag{3}$$

In order to transform the boundary conditions into homogeneous form, let $\tilde{Q}(z, t) = Q(z, t) . Q_1(t)$, then problems (1) can be written as follows:

$$\begin{cases} \dfrac{\partial \tilde{Q}}{\partial t} - \dfrac{\partial}{\partial z}\left( D(Q) \dfrac{\partial \tilde{Q}}{\partial z} \right) = S_r - \dfrac{\partial K(Q)}{\partial z} - \dfrac{\partial Q_1(t)}{\partial t}, & z \in I,\ t \in (0, T) \\ \tilde{Q}(0, t) = 0, \dfrac{\partial \tilde{Q}}{\partial z}(L, t) = 0, & t \in (0, T) \\ \tilde{Q}(z, 0) = Q_0(z) - Q_1(0), & z \in I \end{cases} \tag{4}$$

Let $L^2(I)$ denote the Lebesgue's space of square integrable function on $I$ and $H_1(I)$ a Sobolev space, up to the first derivatives of which are square integrable on $I$. Set $H_E^1(I) = \{ v \in H^1(I);\ v(0) = 0 \}$, $(\cdot, \cdot)$ represents the $L^2$ - inner product on $I$, i. e.:

$$(u, v) = \int_I u(z) v(z)\, \mathrm{d}z, \ \forall u, v \in L^2(I)$$

Let's define, for any $w \in H^1(I)$ and $u, v \in H_E^1(I)$, a bilinear form:

$$D(w; u, v) = \int_I D(w) \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \mathrm{d}z \tag{5}$$

The variational formulation for problem (4) can be written:

Find $\tilde{Q}(z, t): [0, T] \rightarrow H_E^1(I)$, $\forall t \in (0, T)$, such that:

$$\begin{cases} \left(\frac{\partial \widetilde{Q}}{\partial t}, v\right) + D(Q; \widetilde{Q}, v) = \left(S_r - \frac{\partial K(Q)}{\partial z} - \frac{\partial Q_1(t)}{\partial t}, v\right), & \forall v \in H_E^1(I) \\ \widetilde{Q}(z, 0) = Q_0(z) - Q_1(0), & 0 \le z \le L \end{cases} \tag{6}$$

It can be proved that (6) has a unique generalized solution $\widetilde{Q}$.

In order to find the numerical solution for problem (1) or (4), it is necessary to discretize problem (6). We first discretize spatial variable. $S_h = \{I_i; I_i = [z_{i-1}, z_i], i = 1, \cdots, l\}$, $I = \bigcup_{i=1}^{l} I_i$, and $S_h^* = \{I_i^*; I_i^* = [z_{i-\frac{1}{2}}, z_{i+\frac{1}{2}}], i = 1, \cdots, l-1, I_l^* = [z_{l-\frac{1}{2}}, z_l]\}$ denote the primal partition and its dual partition, respectively. Let $h_i = z_i - z_{i-1}$, $h = \max\{h_i; 1 \le i \le l\}$. The partitions are assumed to be regular, that is, there exists a constant $\mu > 0$ independent of h such that $h_i \ge \mu h$, $i = 1, 2, \cdots, l$ The trial function space $U_h = \mathrm{span}\{\phi_1, \cdots, \phi_l\} \subset H_E^1(I) \cap C(I)$ is defined as a piecewise linear function space over $S_h$, here basic function $\phi_i(z)$ defined by:

$$\phi_i(z) = \begin{cases} \frac{z - z_{i-1}}{h_i}, & z \in I_i, \\ \frac{z_{i+1} - z}{h_{i+1}}, & z \in I_{i+1}, \\ 0, & z \in I / (I_i \cup I_{i+1}, \end{cases} \qquad i = 1, \cdots, l-1$$

$$\phi_l(z) = \begin{cases} \frac{z - z_{l-1}}{h_l}, & z \in I_l \\ 0, & z \in I / I_l \end{cases}$$

Any $u_h \in U_h$ can be expressed by $u_h(z) = \sum_{i=1}^{l} u_i \phi_i(z)$, $z \in I$, where $u_i = u_h(z_i)$.

The test function space $v_h = \mathrm{span}(\phi_0, \cdots, \phi_l) \subset L^\infty(z)$ is defined as a piecewise constant function space over $S_h^*$, here basic function $\psi_i(z)$ defined by:

$$\psi_i(z) = \begin{cases} 1, & z \in I_i^*, \\ 0, & z \in I / I_i^*, \end{cases} \qquad i = 1, \cdots, l-1$$

$$\psi_l(z) = \begin{cases} 1, & z \in I_l^* \\ 0, & z \in I / I_l^* \end{cases}$$

Any $v_h \in V_h$ can be expressed by $v_h(z) = \sum_{i=1}^{l} v_i \psi_i(z)$, $z \in I$, where $v_i = v_h(z_i)$.

Obviously, $\dim U_h = \dim V_h = l$

Let's define, for $\forall w \in H^1(I)$, $u \in H_E^1(I)$ and $v_h \in V_h$, a bilinear form:

$$D^*(w; u, v_h) = \sum_{j=1}^{l} v_j D^*(w; u, \psi_j) \tag{7}$$

where $D^*(w; u, \psi_j) = D_{j-\frac{1}{2}} \frac{u_j - u_{j-1}}{h_j} - D_{j+\frac{1}{2}} \frac{u_{j+1} - u_j}{h_{j+1}}$ $u_j = u(z_j)$, $v_j = v_h(z_j)$, $D_{j-\frac{1}{2}} = D(w(z_{j-\frac{1}{2}}, t))$, $D_{l+\frac{1}{2}} = 0$.

We next introduce the generalized Ritz projection operator $R_h^* = R_h^*(t): H_E^1(I) \to U_h$, $0 \le t \le T$ defined by, for $u \in H_E^1(I)$:

$$D^*(Q; u - R_h^* u, v_h) = 0, \ \forall v_h \in V_h \tag{8}$$

where $Q = \widetilde{Q} + Q_1$, $\widetilde{Q}$ is the generalized solution of (6).

Then, the semi-discrete FVM approximation scheme of (4) is to find a map $\widetilde{Q}_h(t): [0, T] \to U_h$ such that:

$$\begin{cases} \left(\frac{\partial \widetilde{Q}_h}{\partial t}, v_h\right) + D^*(Q_h; \widetilde{Q}_h, v_h) = \left(S_r - \frac{\partial K(Q_h)}{\partial z} - \frac{\partial Q_1(t)}{\partial t}, v_h\right), & \forall v_h \in V_h \\ \widetilde{Q}_h(z, 0) = R_h^* Q_0(z) - Q_1(0), & z \in I \end{cases} \tag{9}$$

## 2 Fully discrete finite volume element schemes and numerical examples

Let $N$ be an integer, $\tau = T/N$ be the step length of time, $t_n = n\tau\ (0\le n\le N)$, and $\widetilde{Q}_h^n \in U_h$ be the generalized dierence approximation to $\widetilde{Q}(t_n) \equiv \widetilde{Q}^n$, then the fully discrete FVM approximation scheme of (4) is to seed $\widetilde{Q}_h^{n+1} \le U_h\ (0\le n\le N-1)$, such that:

$$\begin{cases} (\widetilde{Q}_h^{n+1}, v_h) + \tau D^*(Q_h^n; \widetilde{Q}_h^{n+1}, v_h) = \tau\left( S_r^{n+1} - \dfrac{\partial K(Q_h^n)}{\partial z} - \dfrac{\partial Q_1(t_{n+1})}{\partial t}, v_h\right) + (\widetilde{Q}_h^n, v_h),\ \forall v_h \in V_h \\ \widetilde{Q}_h^0 = R_h^* Q_0(z) - Q_1(0), \quad 0 \le z \le L \end{cases} \tag{10}$$

In this section, some numerical examples of the unsaturated soil water flow are given. Without lose of generality, we just take $S_r = 0$ as well, and let $h_i \equiv h = \dfrac{L}{l}$.

In the land surface and atmospheric circulation model, global soil are typically classified and the types of soil parameters are assigned. According to Dickinson et al's BATS model documentation, the parameters for the twelve types of soils are listed in table 1.

**Table 1 Soil parameters of 12 types of soils**

| Soil type/ Parameters | $Q_s$ | $-\psi_s$ / (mm) | $K_s$ / (mm s$^{-1}$) | $b$ | $Q_r/Q_s$ |
|---|---|---|---|---|---|
| 1 | 0.33 | 30 | 0.2000 | 3.5 | 0.088 |
| 2 | 0.36 | 30 | 0.0800 | 4.0 | 0.119 |
| 3 | 0.39 | 30 | 0.0320 | 4.5 | 0.151 |
| 4 | 0.42 | 200 | 0.0130 | 5.0 | 0.266 |
| 5 | 0.45 | 200 | $8.9\times10^{-3}$ | 5.5 | 0.300 |
| 6 | 0.48 | 200 | $6.3\times10^{-3}$ | 6.0 | 0.332 |
| 7 | 0.51 | 200 | $4.5\times10^{-3}$ | 6.8 | 0.378 |
| 8 | 0.54 | 200 | $3.2\times10^{-3}$ | 7.6 | 0.419 |
| 9 | 0.57 | 200 | $2.2\times10^{-3}$ | 8.4 | 0.455 |
| 10 | 0.60 | 200 | $1.6\times10^{-3}$ | 9.2 | 0.487 |
| 11 | 0.63 | 200 | $1.1\times10^{-3}$ | 10.0 | 0.516 |
| 12 | 0.66 | 200 | $0.8\times10^{-3}$ | 10.8 | 0.542 |

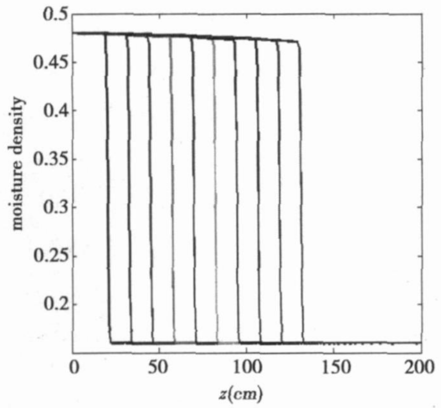
The soil parameters of the sixth type of soil in table 1 are used as examples of the numerical simulation of the finite volume element solution for the infiltration and evaporation.

The soil parameters of the sixth type of soil: $Q_s = 0.48$, $\psi_s = -200$ mm, $K_s = 6.3\times10^{-3}$ (mm/s), $b = 6.0$, $Q_r/Q_s = 0.332$. Taking $L = 200$ cm, the time step length $\tau = 0.2$h, the spatial step size $h = 1$cm, divide the domain $\bar{I} = [0, 200]$ into 200 equal-length elements. Assuming that the surface water flux exceeds the infiltration intensity and runo is generated for some time, a saturated moisture density $Q_1(t) \equiv 0.48$ is maintained at the soil surface ($z = 0$) during the 20h since the beginning of process. And assuming that after 20h, evaporation begins, the moisture density at soil surface rapidly reaches the air-dried moisture density rate and soil at surface keeps the air-dried rate in time interval (26h, 40h], then, the initial and boundary conditions for infiltration and evaporation of water in the soil are:

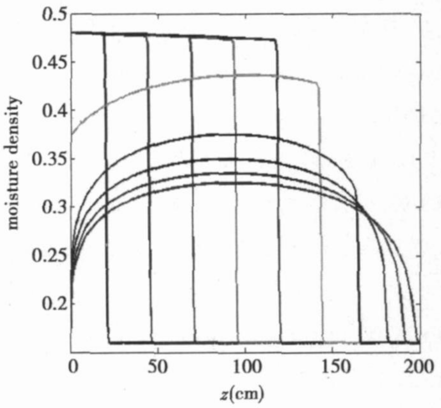
$$Q(z,0)=\begin{cases}0.48\times0.332+0.42\times(1-0.332)\times\frac{10-z}{10}, & z\in[0,10]\\ 0.48\times0.332, & z\in[10,200]\end{cases}$$

$$Q(0,t)=\begin{cases}0.48, & 0<t\leq20\text{h}\\ 0.48+0.48\times(1.0-0.332)\times\frac{20-t}{6}, & 20\text{h}<t\leq26\text{h}\\ 0.48\times0.332, & 26\text{h}<t\leq40\text{h}\end{cases}$$

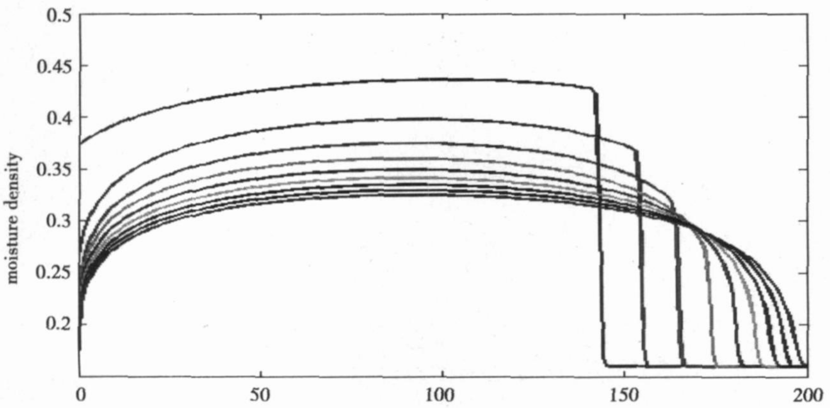
When $S_r=0$, applying the above data, we can obtain the sixth soil moisture density profiles from 0 to 25.5 h, which are shown in fig 1. Fig 1 shows that the moisture density close to surface increases rapidly since infiltration occurred and is gradually close to the saturated soil moisture. From fig 1, we can know that when evaporation occurs, the soil moisture density decreases rapidly with the increase of time. But because of gravity, the moisture density close to the lower of the column will go on to increase. After evaporation occuring for certain time, the curves of soil moisture will change more and more gently.




(a) At each 2h for infiltration




(b) At each 4h for infiltration and evaporation




(c) At each 2h for evaporation

**Fig. 1 The sixth soil moisture density profile**

## 3 Conclusion

From the above analysis, we can find that the results of numerical simulation coincide with the actual situation. Moreover, the finite volume element schemes in this paper are stable and practical. Therefore, it is reliable to solve unsaturated soil moisture density by the schemes in this paper, and we may apply them to numerically simulate more complex physical processes of unsaturated soil water infiltration and evaporation.

**References:**

[1] YE D Z, ZENG Q C, GUO Y F. Model Climate Studies (in Chinese) [M]. Beijing: Climate Press, 1991

[2] DAI Y J, ZENG Q C. A land surface model (IAP94) for climate studies Part I: formulation and validation in o-line experiments [J]. Advances in Atmospheric Sciences, 1997, 14: 433-460

[3] XIE Z H, ZENG Q C, DAI Y J, et al. Researches of numerical simulation of the unsaturated soil water problem [J]. Science in China, Ser D (in Chinese), 1998, 28 (2): 175-180

[4] LI Q. Generalized dierence method [A]. Lecture Notes of the Twelfth Mathematical Workshop [C]. Taejon, Korea, 1997

[5] XIE Z H, ZENG Q C, DAI Y J, et al. An application of the mass-lumped finite element method to the unsaturated soil water problem [J]. Climate and Environment Research (in Chinese), 1998, 28: 73-81

[6] LEI Z D, XANG S X, XIE S C. Soil Hydrology (in Chinese) [M]. Beijing: Tsinghua University Press, 1998

[7] LI H R, LI Q. Finite volume element methods for nonlinear parabolic integrodierential problems [J]. J. KSIAM, 2003, 17 (2): 35-49

[8] LI H R, LUO Z D, LI Q. Generalized Dierence Methods and Numerical Simulation for TWO-dimensional Viscoelastic Problems [J]. Math Num Sin, 2007, 29 (3): 251-262

# 水流入渗问题的数值模拟

李 焕 荣

(重庆工商大学 数学与统计学院,重庆 400067)

**摘　要**:用有限体积元方法建立了非饱和土壤水分入渗问题的数值模型. 通过数值算例比较验证了该数值方法的有效性,同时也得到一些重要的有价值的结论.

**关键词**:数值模拟;水流问题;入渗问题

责任编辑:李翠薇