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加法噪声驱动的随机 Lorenz 系统吸引子及其上半连续性

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摘　要: 确定的 Lorenz 系统是描述大气运动规律的重要数学模型,具有深厚的应用背景,被许多学者广泛研究,然而气候环境受突变因素影响,确定的情形无法完全解释大气的运动规律性;基于此,研究了一种基于加法白噪声驱动的随机 Lorenz 系统的渐进行为,通过恰当的估计证明了系统在参数不受约束条件下存在随机吸收集,进而获得了随机 Lorenz 系统吸引子的存在性,验证了扰动参数趋于零时,随机 Lorenz 系统收敛到确定的系统,从而利用上半连续性的相关理论证明了随机吸引子在 Hausdorff 半距离意义下收敛到全局吸引子,表明 Lorenz 系统的稳定性不受环境因素,比如海啸、地震等的影响。

关键词:随机动力系统;随机 Lorenz 方程组;随机吸引子;上半连续性

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0 引　言

Lorenz 系统是由两无限平板间热对流模型推导出的大气流体动力学模型[1],其被广泛研究[2-3]。Lorenz 吸引子是确定性混沌的一个例子,Stewart[4]首次通过严密的推论证明了非随机情形吸引子的存在,在此之前只能通过计算机数值模拟近似产生,随后全局吸引子的存在性由 Robinson[5] 和 Temam[6] 获得。Robinson 通过建立中心在 z 轴上足够大的球体证明了非随机情形的全局吸引子的存在性,较 Temam 对系统参数加以限制的证明更巧妙。然而,实际大气流体受不确定性因素影响,比如海啸、地震等,非随机情形的 Lorenz 系统无法完全描述这些偶然因素对大气运动规律性的影响。

关于随机 Lorenz 系统的研究由文献[7-8]提出,文献[7]介绍了在解指数稳定的条件下,随机 Lorenz Stenflo 系统的全局吸引集的估计与离散分岔行为。最近,SchmallfuB[8]研究了乘法噪声下 Lorenz 系统吸引子的存在性和维数估计。乘法噪声仅仅增加了乘积因子,没有改变方程的结构,而在加法扰动下,方程的形式显著变化,产生了更多的扰动项,方程的结构更复杂。到目前为止,带加法白噪声的随机 Lorenz 系统吸引子的存在性和上半连续性问题仍然未知。

考虑如下 Lorenz 系统在加法噪声下的随机动力行为:

$$\begin{cases} dx=(-\sigma x+\sigma y)dt+\varepsilon dW_1 \\ dy=(rx-y-xz)dt+\varepsilon dW_2 \\ dz=(xy-bz)dt+\varepsilon dW_3 \end{cases} \tag{1}$$

初始条件:$x(0)=x_0$, $y(0)=y_0$, $z(0)=z_0$。在式(1)中,3 个变量 x,y,z 分别表示温度、湿度和压力;σ,

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b,r 都为正的实参数，σ 表示普朗特数，r 为瑞利数；dW_1,dW_2,dW_3 代表白噪声；$W(t,\omega)=(W_1(t,\omega),W_2(t,\omega),W_3(t,\omega),)$ 为概率空间 (Ω,F,P) 上的双边实值 Wiener 过程，具体形式在后面给出；$\varepsilon\geqslant 0$ 为噪声强度。本文拟利用文献[9]定理 3.1 关于上半连续的结果去研究随机扰动情况下的 Lorenz 系统随机吸引子的存在性及上半连续性等。在对解的估计过程中，Ornstein–Uhlenbeck 的遍历性具有重要作用，所得结果没有对系统的参数给以其他限制[6]。关于随机吸引子的相关理论，读者可参加文献[10–11]等。

文章的结构如下：第一部分通过变量代换把随机 Lorenz 方程组转换成含随机参数的确定性方程组；第二部分证明了随机 Lorenz 方程组吸引子的存在性；第三部分先验证了方程组解的积分具有有界性，接着通过不等式估计和 Gronwall 引理等，研究了当 $\varepsilon\to 0$ 时，随机 Lorenz 方程组的解收敛到确定的方程的解，最后验证了随机吸引子的上半连续性。

1　随机动力系统

本文涉及的概率空间为三维 Wiener 概率空间 (Ω,F,P)，也就是说

$$\Omega=\left\{\omega(t)=(\omega_1(t),\omega_2(t),\omega_3(t))\in C(\mathbf{R},R^3):\ \omega(0)=0;\lim_{t\to\pm\infty}\frac{\omega(t)}{t}=0\right\}$$

F 是由 Ω 生成的 Borel σ–代数，P 是 (Ω,F,P) 上的 Wiener 测度，把三维的 Wiener 过程 $W(t,\omega)$ 和 Ω 中的连续函数 $\omega(t)=(\omega_1(t),\omega_2(t),\omega_3(t))$ 等同，并定义时间平移 $\theta_t\omega(s)=\omega(s+t)-\omega(t),\omega\in\Omega$, $s,t\in\mathbf{R}$，可以验证 $(\Omega,F,P,(\theta_t)_{t\in\mathbf{R}})$ 为距离动力系统。

现在为了将连续随机动力系统与随机 Lorenz 方程组联系起来，需要将带有随机附加项的方程组变换转化为带有随机参数的确定性方程组。为此考虑 O–U 随机微分方程：

$$d\eta_j+\lambda_j\eta_j dt=d\omega_j(t),\lambda_j>0,j=1,2,3 \qquad (2)$$

可以很容易得到式(2)的解为

$$\eta_j(t)=\eta_j(\theta_t\omega_j)=-\lambda_j\int_{-\infty}^{0}e^{\lambda s}(\theta_t\omega_j)(s)ds,t\in\mathbf{R},\lambda_j>0,j=1,2,3$$

需要满足一定的条件：

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\leqslant\frac{c_0{}^2}{16\varepsilon},c_0=\min\{\sigma,b,1\}$$

用 O–U 过程先对 Lorenz 方程组式(1)作变量代换，令

$$x=X+\varepsilon\eta_1(\theta_t\omega_1)$$
$$y=Y+\varepsilon\eta_2(\theta_t\omega_2)$$
$$z=Z+\varepsilon\eta_3(\theta_t\omega_3)$$

则原方程组改写为

$$\begin{cases}\dfrac{dX}{dt}=-\sigma X+\sigma Y+\varepsilon(\lambda_1-\sigma)\eta_1(\theta_t\omega_1)+\varepsilon\sigma\eta_2(\theta_t\omega_2)\\ \dfrac{dY}{dt}=rY-Y-XZ-\varepsilon X\eta_3(\theta_t\omega_3)-\varepsilon Z\eta_1(\theta_t\omega_1)+\\ \qquad \varepsilon r\eta_1(\theta_t\omega_1)+\varepsilon(\lambda_2-1)\eta_2(\theta_t\omega_2)-\\ \qquad \varepsilon^2\eta_1(\theta_t\omega_1)\eta_3(\theta_t\omega_3)\\ \dfrac{dZ}{dt}=XY-bZ+\varepsilon X\eta_2(\theta_t\omega_2)+\varepsilon Y\eta_1(\theta_t\omega_1)+\\ \qquad \varepsilon(\lambda_3-b)\eta_3(\theta_t\omega_3)+\varepsilon^2\eta_1(\theta_t\omega_1)\eta_2(\theta_t\omega_2)\end{cases} \qquad (3)$$

具有初始值：

$$X(0,0,\omega,X_0)=x_0-\varepsilon\eta_1(\omega_1),Y(0,0,\omega,Y_0)=y_0-\varepsilon\eta_2(\omega_2),Z(0,0,\omega,Z_0)=z_0-\varepsilon\eta_3(\omega_3)$$

定义集合族 Δ 为 R^3 空间中满足如下指数收敛条件的随机集的全体：

$$\lim_{t\to\infty}(x_0^2(\theta_{-t}\omega)+y_0^2(\theta_{-t}\omega)+z_0{}^2(\theta_{-t}\omega))e^{-\frac{c_0}{4}t}=0 \qquad (4)$$

其中，$c_0=\min\{\sigma,b,1\}$。

2　随机吸引子的存在性

本节讨论 Lorenz 系统随机吸引子的存在性。首先通过恰当的估计获得了以 $(0,0,r+\sigma)$ 为球心的吸收集的存在性，其中 $r>0,\sigma>0$ 没有其他限制[4]。

引理 1　设集合族 Δ 由式(4)所定义，$0<\varepsilon\leqslant 1$，$B=\{B(\omega)\}_{\omega\in\Omega}\in\Delta$，则对 P–a. e. $\omega\in\Omega$，$\{x_0(\theta_{-t}\omega),y_0(\theta_{-t}\omega),z_0(\theta_{-t}\omega)\}\in B(\theta_{-t}\omega)$，存在 $T(B,\omega)>0$，使得对所有的 $t\geqslant T(B,\omega)$，有 $x^2(t,0,\theta_{-t}\omega,x_0(\theta_{-t}\omega))+y^2(t,0,\theta_{-t}\omega,y_0(\theta_{-t}\omega))+z^2(t,0,\theta_{-t}\omega,z_0(\theta_{-t}\omega))\leqslant r(\omega)$，其中：

$$r(\omega)=8\varepsilon M_1(\omega)+8\int_{-\infty}^{0}e^{\int_s^0\left(-\frac{c_0}{2}+\frac{4}{c_0}M_1(\theta_r\omega)\right)dr}\times(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_s\omega))ds+2(r+\sigma)^2$$

为有限的随机变量,这里 c_0,c_1 为确定的正常数。

$$M_1(\theta_t\omega)=\sum_{i=1}^{3}|\eta_i(\theta_t\omega_i)|^2$$

$$M_2(\theta_t\omega)=\sum_{i=1}^{3}(|\eta_i(\theta_t\omega_i)|^2+|\eta_i(\theta_t\omega_i)|^4)$$

证明 根据方程组式(3),可得

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(X^2+Y^2+(Z-r-\sigma)^2)=\\&-\sigma X^2-Y^2-bZ(Z-r-\sigma)-\varepsilon XY\eta_3(\theta_t\omega_3)-\\&\varepsilon Y(r+\sigma)\eta_1(\theta_t\omega_1)+\varepsilon X(Z-r-\sigma)\eta_2(\theta_t\omega_2)+\\&\varepsilon X(\lambda_1-\sigma)\eta_1(\theta_t\omega_1)+\varepsilon\sigma X\eta_2(\theta_t\omega_2)+\\&\varepsilon(\lambda_2-1)Y\eta_2(\theta_t\omega_2)-\varepsilon^2 Y\eta_1(\theta_t\omega_1)\eta_3(\theta_t\omega_3)+\\&\varepsilon(\lambda_3-b)(Z-r-\sigma)\eta_3(\theta_t\omega_3)+\varepsilon rY\eta_1(\theta_t\omega_1)+\\&\varepsilon^2(Z-r-\sigma)\eta_1(\theta_t\omega_1)\eta_2(\theta_t\omega_2)\end{aligned}\tag{5}$$

下面对上述不等式右端的每一项给出恰当的估计。注意到假设 $\sigma,r,b>0$,则运用 Young 不等式,式(5)中右边各项分别估计为

$$-bZ(Z-r-\sigma)\leqslant-\frac{1}{2}b(Z-r-\sigma)^2+\frac{1}{2}b(r+\sigma)^2\tag{6}$$

$$-\varepsilon XY\eta_3(\theta_t\omega_3)\leqslant\frac{\sigma}{8}X^2+\frac{2}{\sigma}\varepsilon Y^2\eta_3{}^2(\theta_t\omega_3)\tag{7}$$

$$\begin{aligned}&\varepsilon X(Z-r-\sigma)\eta_2(\theta_t\omega_2)\leqslant\\&\frac{b}{8}(Z-r-\sigma)^2+\frac{2}{b}\varepsilon X^2\eta_2{}^2(\theta_t\omega_2)\end{aligned}\tag{8}$$

$$\begin{aligned}&\varepsilon(\lambda_1-\sigma)X\eta_1(\theta_t\omega_1)+\varepsilon\sigma X\eta_2(\theta_t\omega_2)\leqslant\\&\frac{3\sigma}{8}X^2+\frac{1}{\sigma}\varepsilon(\lambda_1-\sigma)^2\eta_1{}^2(\theta_t\omega_1)+2\varepsilon\sigma^2\eta_2{}^2(\theta_t\omega_2)\end{aligned}\tag{9}$$

$$\begin{aligned}&-\varepsilon(r+\sigma)Y\eta_1(\theta_t\omega_1)+\varepsilon rY\eta_1(\theta_t\omega_1)+\\&\varepsilon(\lambda_2-1)Y\eta_2(\theta_t\omega_2)-\varepsilon^2Y\eta_1(\theta_t\omega_1)\eta_3(\theta_t\omega_3)\leqslant\\&\frac{1}{2}Y^2+2(r+\sigma)^2\varepsilon\eta_1{}^2(\theta_t\omega_1)+2\varepsilon r^2\eta_1{}^2(\theta_t\omega_1)+\\&2\varepsilon(\lambda_2-1)^2\eta_2{}^2(\theta_t\omega_2)+2\varepsilon\eta_1{}^2(\theta_t\omega_1)\eta_3{}^2(\theta_t\omega_3)\end{aligned}\tag{10}$$

$$\begin{aligned}&\varepsilon(\lambda_3-b)(Z-r-\sigma)\eta_3(\theta_t\omega_3)+\\&\varepsilon^2(Z-r-\sigma)\eta_1(\theta_t\omega_1)\eta_2(\theta_t\omega_2)\leqslant\\&\frac{b}{8}(Z-r-\sigma)^2+\frac{4}{b}\varepsilon(\lambda_3-b)^2\eta_3{}^2(\theta_t\omega_3)+\\&\frac{4}{b}\varepsilon\eta_1{}^2(\theta_t\omega_1)\eta_2{}^2(\theta_t\omega_2)\end{aligned}\tag{11}$$

记 $M_1(\theta_t\omega)=\sum_{i=1}^{3}|\eta_i(\theta_t\omega_i)|^2$,$M_2(\theta_t\omega)=\sum_{i=1}^{3}(|\eta_i(\theta_t\omega_i)|^2+|\eta_i(\theta_t\omega_i)|^4)$,结合式(5)—式(11),令 $c_0=\min\{\sigma,b,1\}$,于是有

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(X^2+Y^2+(Z-r-\sigma)^2)\leqslant\\&\left(-\frac{c_0}{2}+\frac{4}{c_0}\varepsilon M_1(\theta_t\omega)\right)X^2+\left(-\frac{c_0}{2}+\frac{4}{c_0}\varepsilon M_1(\theta_t\omega)\right)Y^2-\\&\frac{c_0}{2}(Z-r-\sigma)^2+b(r+\sigma)^2+c_1\varepsilon M_2(\theta_t\omega)\leqslant\\&\left(-\frac{c_0}{2}+\frac{4}{c_0}\varepsilon M_1(\theta_t\omega)\right)(X^2+Y^2+(Z-r-\sigma)^2)+\\&b(r+\sigma)^2+c_1\varepsilon M_2(\theta_t\omega)\end{aligned}\tag{12}$$

其中,c_1 为依赖于 σ,b,r 的确定的正常数。在式(12)中,运用 Gronwall 引理,并令 $s-t=s'$,$r-t=r'$,有

$$\begin{aligned}&X^2(t,0,\theta_{-t}\omega,X_0(\theta_{-t}\omega))+\\&Y^2(t,0,\theta_{-t}\omega,Y_0(\theta_{-t}\omega))+\\&(Z(t,0,\theta_{-t}\omega,Z_0(\theta_{-t}\omega))-r-\sigma)^2\leqslant\\&(X_0{}^2(\theta_{-t}\omega)+Y_0{}^2(\theta_{-t}\omega)+\\&(Z_0(\theta_{-t}\omega)-r-\sigma)^2)\mathrm{e}^{\int_{-t}^{0}\left(-\frac{c_0}{2}+\frac{4}{c_0}M_1(\theta_s\omega)\right)\mathrm{d}s}+\\&\int_{-t}^{0}\mathrm{e}^{\int_{s}^{0}\left(-\frac{c_0}{2}+\frac{4}{c_0}M_1(\theta_r\omega)\right)\mathrm{d}r}(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_s\omega))\mathrm{d}s\end{aligned}$$

取适当 λ_i,$i=1,2,3$,使 $\frac{1}{2\lambda_1}+\frac{1}{2\lambda_2}+\frac{1}{2\lambda_3}\leqslant\frac{c_0{}^2}{32\varepsilon}$,根据 Ornstein-Uhlenbeck 的特征,当 $t\to\infty$ 时,有

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}\left[-\frac{c_0}{2}+\frac{4}{c_0}\varepsilon M_1(\theta_s\omega)\right]\mathrm{d}s=\\&\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}\left[-\frac{c_0}{2}+\frac{4\varepsilon}{c_0}(\eta_1^2(\theta_t\omega_1)+\eta_2^2(\theta_t\omega_2)+\right.\\&\left.\eta_3^2(\theta_t\omega_3))\right]\mathrm{d}s=\\&-\frac{c_0}{2}+\frac{4}{c_0}\varepsilon E(\eta_1{}^2(\omega_1)+\eta_2{}^2(\omega_2)+\eta_3{}^2(\omega_3))=\\&-\frac{c_0}{2}+\frac{4}{c_0}\varepsilon\left(\frac{1}{2\lambda_1}+\frac{1}{2\lambda_2}+\frac{1}{2\lambda_3}\right)\leqslant\\&-\frac{c_0}{2}+\frac{4\varepsilon}{c_0}\times\frac{c_0{}^2}{32\varepsilon}-\leqslant\frac{c_0}{4}\end{aligned}$$

则存在充分大的 T_0,当 $t>T_0$ 时成立,有

$$\int_{-t}^{0}\left[-\frac{c_0}{2}+\frac{4}{c_0}\varepsilon M_1(\theta_s\omega)\right]\mathrm{d}s\leqslant-\frac{c_0}{4}t$$

故当 $t>T_0$ 时,有

$$\begin{aligned}&X^2(t,0,\theta_{-t}\omega,X_0(\theta_{-t}\omega))+\\&Y^2(t,0,\theta_{-t}\omega,Y_0(\theta_{-t}\omega))+\end{aligned}$$

$$(Z(t,0,\theta_{-t}\omega,Z_0(\theta_{-t}\omega))-r-\sigma)^2\leqslant (X_0{}^2(\theta_{-t}\omega)+Y_0{}^2(\theta_{-t}\omega)+(Z_0(\theta_{-t}\omega)-r-\sigma)^2)\mathrm{e}^{-\frac{c_0}{4}t}+\int_{-\infty}^0 \mathrm{e}^{\int_s^0\left(-\frac{c_0}{2}+\frac{4}{c_0}M_1(\theta_r\omega)\right)\mathrm{d}r}(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_s\omega))\mathrm{d}s \quad (13)$$

容易验证式(13)右端第二项中的积分是有限的。由变量代换 $x=X+\varepsilon\eta_1(\theta_t\omega_1)$, $y=Y+\varepsilon\eta_2(\theta_t\omega_2)$, $z=Z+\varepsilon\eta_3(\theta_t\omega_3)$,即有

$$x^2(t,0,\theta_{-t}\omega,x_0(\theta_{-t}\omega))+y^2(t,0,\theta_{-t}\omega,y_0(\theta_{-t}\omega))+(z(t,0,\theta_{-t}\omega,z_0(\theta_{-t}\omega))-r-\sigma)^2\leqslant 4(x_0{}^2(\theta_{-t}\omega)+y_0{}^2(\theta_{-t}\omega)+(z_0(\theta_{-t}\omega)-r-\sigma)^2+M_1(\theta_{-t}\omega))\mathrm{e}^{-\frac{c_0}{4}t}+2\varepsilon M_1(\omega)+2\int_{-\infty}^0 \mathrm{e}^{\int_s^0\left(-\frac{c_0}{2}+\frac{4}{c_0}M_1(\theta_r\omega)\right)\mathrm{d}r}\times(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_s\omega))\mathrm{d}s$$

因为 $\{x_0(\theta_{-t}\omega),y_0(\theta_{-t}\omega),z_0(\theta_{-t}\omega)\}\in B(\theta_{-t}\omega)$ 和 $M_1(\theta_{-t}\omega)$ 关于 t 最多二次多项式增长,因此存在 $T_1>T_0$,当所有的 $t>T_1$ 时,有

$$x^2(t,0,\theta_{-t}\omega,x_0(\theta_{-t}\omega))+y^2(t,0,\theta_{-t}\omega,y_0(\theta_{-t}\omega))+z^2(t,0,\theta_{-t}\omega,z_0(\theta_{-t}\omega))\leqslant 8\varepsilon M_1(\omega)+8\int_{-\infty}^0 \mathrm{e}^{\int_s^0\left(-\frac{c_0}{2}+\frac{4}{c_0}M_1(\theta_r\omega)\right)\mathrm{d}r}\times(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_s\omega))\mathrm{d}s+2(r+\sigma)^2$$

从而引理 1 得证。

根据方程组式(1)的解算子 $\{x(t,0,\omega,x_0),y(t,0,\omega,y_0),z(t,0,\omega,z_0)\}$ 定义随机动力系统 φ: $\varphi(t,\omega,(x_0,y_0,z_0))=(x(t,0,\omega,x_0),y(t,0,\omega,y_0),z(t,0,\omega,z_0))$。令 $K(\omega)=\{(x,y,z):x^2+y^2+z^2\leqslant r(\omega),\omega\in\Omega\}$,半径 $r(\omega)$ 由引理 1 确定,则有如下存在性结果。

定理 1　对任意的 $\varepsilon\in(0,1]$,由 Lorenz 方程组式(1)确定的随机动力系统 φ 存在唯一的 Δ-随机吸引子 $\{A(\omega)\}_{\omega\in\Omega}$,其中

$$A(\omega)=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\varphi(t,\theta_{-t}\omega,K(\theta_{-t}\omega))},\omega\in\Omega$$

证明　由引理 1,对 $\omega\in\Omega$ 及任意的 $x\in R^3$,有

$$\mathrm{dist}(x,K(\omega))=\begin{cases}0, & \text{当 } x\in K(\omega)\\ |x|-\sqrt{r(\omega)}, & \text{当 } x\notin K(\omega)\end{cases}$$

故映射 $\omega\longmapsto \mathrm{dist}(x,K(\omega))$ 依 F 可测,因此 $\{K(\omega)\}_{\omega\in\Omega}$ 为随机吸收集。另一方面,需要证明 $\lim\limits_{t\to\infty}\mathrm{e}^{-\frac{c_0}{4}t}r(\theta_{-t}\omega)=0$。首先估计 $r(\theta_{-t}\omega)$ 中的积分项的收敛性:

$$\int_{-\infty}^0 \mathrm{e}^{\int_s^0\left(-\frac{c_0}{2}+\frac{4}{c_0}\varepsilon M_1(\theta_{r-t}\omega)\right)\mathrm{d}r}(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_{s-t}\omega))\mathrm{d}s=\int_{-\infty}^{-t}\mathrm{e}^{\frac{c_0(t+s)}{2}+\int_s^{-t}\left(\frac{4}{c_0}\varepsilon M_1(\theta_r\omega)\right)\mathrm{d}r}(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_s\omega))\mathrm{d}s\leqslant\int_{-\infty}^{-t}\mathrm{e}^{\frac{c_0(t+s)}{4}+\int_s^0\left(\frac{4}{c_0}\varepsilon M_1(\theta_r\omega)\right)\mathrm{d}r}(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_s\omega))\mathrm{d}s=\mathrm{e}^{\frac{c_0}{4}t}\int_{-\infty}^{-t}\mathrm{e}^{\int_s^0\left(-\frac{c_0}{4}+\frac{4}{c_0}\varepsilon M_1(\theta_r\omega)\right)\mathrm{d}r}(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_s\omega))\mathrm{d}s$$

根据假设条件 $\frac{1}{2\lambda_1}+\frac{1}{2\lambda_2}+\frac{1}{2\lambda_3}\leqslant\frac{c_0{}^2}{32\varepsilon}$,有

$$\lim_{s\to-\infty}\frac{1}{-s}\int_s^0\left[-\frac{c_0}{4}+\frac{4}{c_0}\varepsilon M_1(\theta_s\omega)\right]\mathrm{d}s=-\frac{c_0}{4}+\frac{4}{c_0}\varepsilon\left(\frac{1}{2\lambda_1}+\frac{1}{2\lambda_2}+\frac{1}{2\lambda_3}\right)\leqslant-\frac{c_0}{8}$$

于是当 $s\to-\infty$ 时, $\int_s^0\left[-\frac{c_0}{4}+\frac{4}{c_0}\varepsilon M_1(\theta_s\omega)\right]\mathrm{d}s\leqslant\frac{c_0}{8}s$。又因为 $M_2(\theta_s\omega)$ 关于 s 最多四次多项式增长,所以反常积分: $\int_{-\infty}^0 \mathrm{e}^{\int_s^0\left(-\frac{c_0}{4}+\frac{4}{c_0}\varepsilon M_1(\theta_r\omega)\right)\mathrm{d}r}(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_s\omega))\mathrm{d}s<+\infty$ 收敛,从而可得

$$\lim_{t\to\infty}\mathrm{e}^{-\frac{c_0}{4}t}\int_{-\infty}^0 \mathrm{e}^{\int_s^0\left(-\frac{c_0}{2}+\frac{4}{c_0}\varepsilon M_1(\theta_r\omega)\right)\mathrm{d}r}\times(b(r+\sigma)^2+c_1\varepsilon M_2(\theta_{s-t}\omega))\mathrm{d}s\leqslant 0$$

于是 $\lim\limits_{t\to\infty}\mathrm{e}^{-\frac{c_0}{4}t}r(\theta_{-t}\omega)=0$,即 $\{K(\omega)\}_{\omega\in\Omega}$ 为闭的有界 Δ-随机吸收集。

3　随机吸引子的上半连续性

为了获得随机吸引子的上半连续,需要证明随机 Lorenz 方程组的解关于参数 ε 的收敛结果,为此预先证明解的积分有限。

引理 2　设 $\varepsilon\in(0,1]$, $\omega\in\Omega$, $T>0$, $(X^\varepsilon,Y^\varepsilon,Z^\varepsilon)$ 是方程组式(3)当初值为 $(X^\varepsilon(0),Y^\varepsilon(0),Z^\varepsilon(0))$ 时

的解,则存在常数 $c_1(T,\omega)$ 和 $c_2(T,\omega)$,使得对 $\forall t\in(0,T]$,有

$$\int_0^t (X^{\varepsilon 2}(s)+Y^{\varepsilon 2}(s)+Z^{\varepsilon 2}(s))\mathrm{d}s\leqslant c_1(T,\omega)(x^{\varepsilon 2}(0)+y^{\varepsilon 2}(0)+z^{\varepsilon 2}(0))+c_2(T,\omega)$$

其中,$x^{\varepsilon}(0)=X^{\varepsilon}(0)+\varepsilon\eta_1(\omega_1)$,$y^{\varepsilon}(0)=Y^{\varepsilon}(0)+\varepsilon\eta_2(\omega_2)$,$z^{\varepsilon}(0)=Z^{\varepsilon}(0)+\varepsilon\eta_3(\omega_3)$。

证明 由 O-U 过程的连续性可知,对 $\forall T>0$,$\exists N(T,\omega)>0$,使得对 $\forall t\in(0,T]$,成立:

$$\begin{cases}\dfrac{4}{c_0}\left|\int_0^t M_1(\theta_s\omega)\mathrm{d}s\right|\leqslant N(T,\omega)\\ \left|\int_0^t M_2(\theta_s\omega)\mathrm{d}s\right|\leqslant N(T,\omega)\end{cases}\tag{14}$$

由式(13)得到:

$$\frac{\mathrm{d}}{\mathrm{d}t}(X^{\varepsilon 2}+Y^{\varepsilon 2}+(Z^{\varepsilon}-r-\sigma)^2)+\frac{c_0}{2}(X^{\varepsilon 2}+Y^{\varepsilon 2}+(Z^{\varepsilon}-r-\sigma)^2)\leqslant \frac{4}{c_0}M_1(\theta_t\omega)(X^{\varepsilon 2}+Y^{\varepsilon 2}+(Z^{\varepsilon}-r-\sigma)^2)+b(r+\sigma)^2+c_1M_2(\theta_t\omega)\tag{15}$$

将式(15)两端同时乘以 $\mathrm{e}^{-\frac{4}{c_0}\int_0^t M_1(\theta_\tau\omega)\mathrm{d}\tau}$,再从 0 到 t 积分,结合式(14),可得:

$$X^{\varepsilon 2}(t)+Y^{\varepsilon 2}(t)+(Z^{\varepsilon}(t)-r-\sigma)^2+\frac{c_0}{2}\int_0^t \mathrm{e}^{\frac{4}{c_0}\int_s^t M_1(\theta_\tau\omega)\mathrm{d}\tau}[X^{\varepsilon 2}(s)+Y^{\varepsilon 2}(s)+(Z^{\varepsilon}(s)-r-\sigma)^2]\mathrm{d}s\leqslant$$
$$[X^{\varepsilon 2}(0)+Y^{\varepsilon 2}(0)+(Z^{\varepsilon}(0)-r-\sigma)^2]\mathrm{e}^{N(T,\omega)}+\mathrm{e}^{N(T,\omega)}[Tb(r+\sigma)^2+c_1N(T,\omega)]\leqslant$$
$$4[x^{\varepsilon 2}(0)+y^{\varepsilon 2}(0)+z^{\varepsilon 2}(0)+(r+\sigma)^2+M_1(\theta_t\omega)]\times N(T,\omega)+\mathrm{e}^{N(T,\omega)}[Tb(r+\sigma)^2+c_1N(T,\omega)]$$

由于当 $s,t\in(0,T]$ 时,有 $\mathrm{e}^{\frac{4}{c_0}\int_s^t M_1(\theta_\tau\omega)\mathrm{d}\tau}>1$,故

$$\int_0^t (X^{\varepsilon 2}(s)+Y^{\varepsilon 2}(s)+(Z^{\varepsilon}(s)-r-\sigma)^2)\mathrm{d}s\leqslant \frac{8\mathrm{e}^{N(T,\omega)}}{c_0}(x^{\varepsilon 2}(0)+y^{\varepsilon 2}(0)+z^{\varepsilon 2}(0)+(r+\sigma)^2+M_1(\theta_t\omega))+\frac{2\mathrm{e}^{N(T,\omega)}}{c_0}(Tb(r+\sigma)^2+c_1N(T,\omega))\tag{16}$$

注意到 $Z^{\varepsilon 2}(s)\leqslant 2(Z^{\varepsilon}(s)-r-\sigma)^2+2(r+\sigma)^2$,故存在正常数 $c_1(T,\omega)$ 和 $c_2(T,\omega)$,成立:

$$\int_0^t (X^{\varepsilon 2}(s)+Y^{\varepsilon 2}(s)+Z^{\varepsilon 2}(s))\mathrm{d}s\leqslant c_1(T,\omega)(x^{\varepsilon 2}(0)+y^{\varepsilon 2}(0)+z^{\varepsilon 2}(0))+c_2(T,\omega)$$

结果得证。

定理 2 设 $\varepsilon\in(0,1]$,$\omega\in\Omega$,$T>0$,令 $(x^{\varepsilon},y^{\varepsilon},z^{\varepsilon})$ 为方程组式(1)在初始条件 $(x_0^{\varepsilon},y_0^{\varepsilon},z_0^{\varepsilon})$ 下的解;(x,y,z) 是方程组式(1)当 $\varepsilon=0$ 时在初始条件 (x_0,y_0,z_0) 下的解。如果当 $\varepsilon\to 0$,$(x_0^{\varepsilon},y_0^{\varepsilon},z_0^{\varepsilon})\to(x_0,y_0,z_0)$,对 $\forall t\in(0,T]$,有

$$\lim_{\varepsilon\to 0}\{(x^{\varepsilon}(t,0,\omega,x_0^{\varepsilon})-x(t,x_0))^2+(y^{\varepsilon}(t,0,\omega,y_0^{\varepsilon})-y(t,y_0))^2+(z^{\varepsilon}(t,0,\omega,z_0^{\varepsilon})-z(t,z_0))^2\}=0$$

证明 令 $\hat{x}=X^{\varepsilon}-x$,$\hat{y}=Y^{\varepsilon}-y$ 和 $\hat{z}=Z^{\varepsilon}-z$,其中 $(X^{\varepsilon},Y^{\varepsilon},Z^{\varepsilon})$ 是式(3)的解,则 $(\hat{x},\hat{y},\hat{z})$ 满足如下方程:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{x}^2+\hat{y}^2+\hat{z}^2)=&-\sigma\hat{x}^2-\hat{y}^2-b\hat{z}^2+\sigma\hat{x}\hat{y}+r\hat{x}\hat{y}+\hat{x}(Y^{\varepsilon}\hat{z}-Z^{\varepsilon}\hat{y})+\varepsilon\hat{x}(\lambda_1-\sigma)\eta_1(\theta_t\omega_1)+\\&\varepsilon\sigma\hat{x}\eta_2(\theta_t\omega_2)-\varepsilon X^{\varepsilon}\hat{y}\eta_3(\theta_t\omega_3)-\varepsilon Z^{\varepsilon}\hat{y}\eta_1(\theta_t\omega_1)+\\&\varepsilon Y^{\varepsilon}\hat{z}\eta_1(\theta_t\omega_1)+\varepsilon X^{\varepsilon}\hat{z}\eta_2(\theta_t\omega_2)+\hat{y}(r\varepsilon\eta_1(\theta_t\omega_1)+\\&\varepsilon(\lambda_2-1)\eta_2(\theta_t\omega_2)-\varepsilon^2\eta_1(\theta_t\omega_1)\eta_3(\theta_t\omega_3))+\\&\hat{z}(\varepsilon(\lambda_3-b)\eta_3(\theta_t\omega_3)+\varepsilon^2\eta_1(\theta_t\omega_1)\eta_2(\theta_t\omega_2))\end{aligned}\tag{17}$$

运用 Young 不等式,对式(17)右端各项分别估计,下面记 c 为通用常数,有

$$\begin{cases}\sigma\hat{x}\hat{y}\leqslant\dfrac{\sigma}{8}\hat{x}^2+2\sigma\hat{y}^2,\ r\hat{x}\hat{y}\leqslant 2r^2\hat{x}^2+\dfrac{1}{8}\hat{y}^2\\ \varepsilon\hat{x}(\lambda_1-\sigma)\eta_1(\theta_t\omega_1)\leqslant\dfrac{\sigma}{8}\hat{x}^2+c\varepsilon\eta_1^{\ 2}(\theta_t\omega_1)\\ \varepsilon\hat{x}\sigma\eta_2(\theta_t\omega_2)\leqslant\dfrac{\sigma}{8}\hat{x}^2+c\varepsilon\eta_2^{\ 2}(\theta_t\omega_2)\\ \hat{x}(Y^{\varepsilon}\hat{z}-Z^{\varepsilon}\hat{y})\leqslant\dfrac{\sigma}{8}\hat{x}^2+\dfrac{2}{\sigma}Y^{\varepsilon 2}\hat{z}^2+\dfrac{2}{\sigma}Z^{\varepsilon 2}\hat{y}^2\\ -\varepsilon X^{\varepsilon}\hat{y}\eta_3(\theta_t\omega_3)\leqslant\dfrac{1}{8}\hat{y}^2+c\varepsilon X^{\varepsilon 2}\eta_1^{\ 2}(\theta_t\omega_1)\\ -\varepsilon Z^{\varepsilon}\hat{y}\eta_1(\theta_t\omega_1)\leqslant\dfrac{1}{8}\hat{y}^2+c\varepsilon Z^{\varepsilon 2}\eta_1^{\ 2}(\theta_t\omega_1)\\ \varepsilon Y^{\varepsilon}\hat{z}\eta_1(\theta_t\omega_1)\leqslant\dfrac{b}{8}\hat{z}^2+c\varepsilon Y^{\varepsilon 2}\eta_1^{\ 2}(\theta_t\omega_1)\\ \varepsilon X^{\varepsilon}\hat{z}\eta_2(\theta_t\omega_2)\leqslant\dfrac{b}{8}\hat{z}^2+c\varepsilon X^{\varepsilon 2}\eta_2^{\ 2}(\theta_t\omega_2)\end{cases}\tag{18}$$

$$\begin{cases}\hat{y}((\varepsilon r-\varepsilon^2\eta_3(\theta_t\omega_3))\eta_1(\theta_t\omega_1)+\\ \varepsilon(\lambda_2-1)\eta_2(\theta_t\omega_2))\leqslant\frac{1}{8}\hat{y}^2+\\ c\varepsilon(\eta_1{}^2(\theta_t\omega_1)(1+\eta_3{}^2(\theta_t\omega_3))+\eta_2{}^2(\theta_t\omega_2))\\ \hat{z}(\varepsilon(\lambda_3-b)\eta_3(\theta_t\omega_3)+\varepsilon^2\eta_1(\theta_t\omega_1)\eta_2(\theta_t\omega_2))\leqslant\\ \frac{b}{4}\hat{z}^2+c\varepsilon\eta_3{}^2(\theta_t\omega_3)+c\varepsilon\eta_1{}^2(\theta_t\omega_1)\eta_2{}^2(\theta_t\omega_2)\end{cases} \tag{19}$$

于是结合式(17)—式(19),令 $k_0=\max\{2r^2,2\sigma\}$,则有

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{x}^2+\hat{y}^2+\hat{z}^2)\leqslant k_0(\hat{x}^2+\hat{y}^2+\hat{z}^2)+\frac{2}{\sigma}(X^{\varepsilon2}+Y^{\varepsilon2}+Z^{\varepsilon2})(\hat{x}^2+\hat{y}^2+\hat{z}^2)+c\varepsilon(X^{\varepsilon2}+Y^{\varepsilon2}+Z^{\varepsilon2})M_1(\theta_t\omega)+c\varepsilon M_2(\theta_t\omega)$$

由此运用 Gronwall 引理得,有

$$\hat{x}^2+\hat{y}^2+\hat{z}^2\leqslant(\hat{x}(0)^2+\hat{y}(0)^2+\hat{z}(0)^2)\times \mathrm{e}^{\int_0^t[k_0+\frac{2}{\sigma}(X^{\varepsilon2}(s)+Y^{\varepsilon2}(s)+Z^{\varepsilon2}(s))]\,\mathrm{d}s}+\int_0^t \mathrm{e}^{\int_s^t[k_0+\frac{2}{\sigma}(X^{\varepsilon2}(\tau)+Y^{\varepsilon2}(\tau)+Z^{\varepsilon2}(\tau))]\,\mathrm{d}\tau}\times c\varepsilon(X^{\varepsilon2}(s)+Y^{\varepsilon2}(s)+Z^{\varepsilon2}(s))M_1(\theta_s\omega)\mathrm{d}s+\int_0^t \mathrm{e}^{\int_s^t[k_0+\frac{2}{\sigma}(X^{\varepsilon2}(\tau)+Y^{\varepsilon2}(\tau)+Z^{\varepsilon2}(\tau))]\,\mathrm{d}\tau}c\varepsilon M_2(\theta_s\omega)\mathrm{d}s$$

再次利用 O-U 过程的连续性,$\exists N_1(T,\omega)$,使得对 $t\in(0,T]$,$M_1(\theta_t\omega)\leqslant N_1(T,\omega)$,$M_2(\theta_t\omega)\leqslant N_1(T,\omega)$,由引理 1 的结果,可得:

$$\hat{x}^2+\hat{y}^2+\hat{z}^2\leqslant(\hat{x}(0)^2+\hat{y}(0)^2+\hat{z}(0)^2)\times \mathrm{e}^{Tk_0+\frac{2\bar{c}}{\sigma}}+c\varepsilon e^{Tk_0+\frac{2\bar{c}}{\sigma}}(N_1(T,\omega)(\bar{c}+T))\leqslant c_1(T,\omega)(\hat{x}(0)^2+\hat{y}(0)^2+\hat{z}(0)^2)+c\varepsilon c_2(T,\omega) \tag{20}$$

$\bar{c}$ 由引理 1 给出,其中 $c_1(T,\omega)=\mathrm{e}^{Tk_0+\frac{2}{\sigma}\bar{c}}$,$c_2(T,\omega)=(c+T)\mathrm{e}^{Tk_0+\frac{2}{\sigma}\bar{c}}N_1(T,\omega)$,利用

$$\hat{x}=X^{\varepsilon}-x=(x^{\varepsilon}-x)+\eta_1(\theta_t\omega_1)$$

$$\hat{y}=Y^{\varepsilon}-y=(y^{\varepsilon}-y)+\eta_2(\theta_t\omega_2)$$

$$\hat{z}=Z^{\varepsilon}-z=(z^{\varepsilon}-z)+\eta_3(\theta_t\omega_3)$$

由式(20)可得:

$$(x^{\varepsilon}(t)-x)^2+(y^{\varepsilon}(t)-y)^2+(z^{\varepsilon}(t)-z)^2\leqslant 2(\hat{x}^2+\hat{y}^2+\hat{z}^2)+2\varepsilon M_1(\theta_t\omega)\leqslant 2c_1(T,\omega)((x_0^{\varepsilon}-x_0)^2+(y_0^{\varepsilon}-y_0)^2+(z_0^{\varepsilon}-z_0)^2)+c\varepsilon c_2(T,\omega)$$

因此 $\varepsilon\to0$,$x_0^{\varepsilon}-x_0\to0$,$y_0^{\varepsilon}-y_0\to0$,$z_0^{\varepsilon}-z_0\to0$,对所有的 $t\in(0,T]$,有

$$(x^{\varepsilon}(t)-x(t))^2+(y^{\varepsilon}(t)-y(t))^2+(z^{\varepsilon}(t)-z(t))^2\to0$$

从而引理 2 得证。

定理 3　设 $\{A_{\varepsilon}(\omega)\}_{\omega\in\Omega}$ 是方程式(1)生成的随机动力系统 φ_{ε} 的随机吸引子,A_0 是方程组式(1)当 $\varepsilon=0$ 时所得的确定性动力系统 φ_0 的全局吸引子,则对 P-a. e. $\omega\in\Omega$ 成立,有

$$\lim_{\varepsilon\to0}\mathrm{dist}\,(A_{\varepsilon}(\omega),A_0)=0$$

证明　由定理 2 解的收敛性可知,对 P-a. e. $\omega\in\Omega$,当 $\varepsilon\to0$,$(x_0^{\varepsilon},y_0^{\varepsilon},z_0^{\varepsilon})\to(x_0,y_0,z_0)$ 时,有

$$\lim_{\varepsilon\to0}\varphi_{\varepsilon}(t,\omega,(x_0^{\varepsilon},y_0^{\varepsilon},z_0^{\varepsilon}))=\varphi_0(t,(x_0,y_0,z_0))$$

成立,根据引理 1,有

$$\lim_{\varepsilon\to0}\sup\|E_{\varepsilon}(\omega)\|\leqslant\left(\frac{8b\,(r+\sigma)^2}{c_0}+2\,(r+\sigma)^2\right)^{\frac{1}{2}}$$

这里的 $E_{\varepsilon}(\omega)$ 是 φ_{ε} 的随机吸收集。显然 $\bigcup_{0<\varepsilon\leqslant1}A_{\varepsilon}(\omega)$ 在 R^3 中是准紧的,故上半连续性得证。

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Attractors and Their Upper Semi-continuity of Stochastic Lorenz System Driven by Additive Noises

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Abstract: The deterministic Lorenz system is an important mathematical model for describing the law of atmospheric motion. It has a profound application background and has been extensively studied by many scholars. However, the climate environment is affected by abrupt factors, and the determined situation cannot fully explain the regularity of atmospheric motion. Based on this, in this paper, we study the progressive behavior of a stochastic Lorenz system driven by additive white noise. We firstly obtain the existence of the random attractor of the random Lorenz system by properly estimating the existence of random absorption set under the condition that the parameters are not constrained. We secondly prove that the stochastic Lorenz system converges to a deterministic system when the perturbation parameter tends to zero. Thirdly, by the theory of the upper semicontinuity, we obtain that random attractor converges to the global attractor in the sense of Hausdorff half-distance. This indicates that the stability of the Lorenz system is not affected by environmental factors such as tsunamis, earthquakes, etc.

Key words: stochastic dynamical system; stochastic Lorenz system; random attractor; upper semicontinuity

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