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多元函数极值新的判定方法

赵　坤
(山东科技大学 数学与系统科学学院,山东 青岛 266590)

摘　要:函数的极值有重要的研究意义,求解方法多种多样;以三元函数一般的正定性判定方法为根据,得到了一种新的三元函数极值判定方法及证明过程,这种方法适用于条件和非条件极值的情况,并将这种判定方法推广到多元函数,得到一种多元函数极值判定方法.

关键词:极值;三元函数;多元函数;判定方法
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通过对高等数学的学习和研究,发现对于一元、二元函数极值的判定方法有了比较充分的研究[1-3],但是三元及多元函数极值的判定方法不同于二元函数极值判定方法,对于三元及多元函数的判定方法研究较少,有 Jacobi 矩阵法[4]和梯度法[5]等.文章根据文献[6-8]研究了三元函数极值新的判定方法,依据文献[9]推广得到了多元函数极值新的判定方法.

1　三元函数极值新的判定方法及证明

定理 1　若三元函数 $f(x,y,z)$ 存在驻点 (x_0,y_0,z_0),且在驻点处有二阶连续偏导数,令 $a_{11}=f_{x^2}(x_0,y_0,z_0)$, $a_{12}=f_{xy}(x_0,y_0,z_0)$, $a_{13}=f_{xz}(x_0,y_0,z_0)$, $a_{22}=f_{y^2}(x_0,y_0,z_0)$, $a_{23}=f_{yz}(x_0,y_0,z_0)$, $a_{33}=f_{z^2}(x_0,y_0,z_0)$,则当满足 $\begin{cases}a_{11}-a_{12}^2-a_{13}^2>0\\ a_{22}-a_{23}^2-1>0\\ a_{33}-2>0\end{cases}$ 时,函数 $f(x,y,z)$ 在点 (x_0,y_0,z_0) 处取极小值;当满足 $\begin{cases}a_{11}+a_{12}^2+a_{13}^2<0\\ a_{22}+a_{23}^2+1<0\\ a_{33}+2<0\end{cases}$ 时,函数 $f(x,y,z)$ 在点 (x_0,y_0,z_0) 处取极大值.

证明　由三元函数的泰勒展开式且 $f_x=f_y=f_z=0$,就有

$$\Delta f=f(x_0+\Delta x,y_0+\Delta y,z_0+\Delta z)-f(x_0,y_0,z_0)=$$

$$\frac{1}{1!}(f_x\Delta x+f_y\Delta y+f_z\Delta z)+\frac{1}{2!}(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y+\frac{\partial}{\partial z}\Delta z)^2f(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)=$$

$$\frac{1}{2}[f_{x^2}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)\Delta x^2+2f_{xy}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)\Delta x\Delta y+$$

$$2f_{xz}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)\Delta x\Delta z+f_{y^2}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)\Delta y^2+$$

$$2f_{yz}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)\Delta y\Delta z+f_{z^2}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)\Delta z^2]$$

由于三元函数的一切二阶偏导数在点 (x_0,y_0,z_0) 处连续,由已知条件

$$f_{x^2}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)=a_{11}+\alpha_{11},\quad f_{xy}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)=a_{12}+\alpha_{12}$$

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作者简介:赵坤(1990-),女,山东潍坊人,硕士研究生,从事微分方程及其应用研究.

$$f_{xz}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)=a_{13}+\alpha_{13},\quad f_{y^2}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)=a_{22}+\alpha_{22}$$

$$f_{yz}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)=a_{23}+\alpha_{23},\quad f_{z^2}(x_0+\theta\Delta x,y_0+\theta\Delta y,z_0+\theta\Delta z)=a_{33}+\alpha_{33}$$

其中 $\alpha_{11},\alpha_{12},\alpha_{13},\alpha_{22},\alpha_{23},\alpha_{33}$ 是无穷小量,于是得到

$$\begin{aligned}\Delta f=&\frac{1}{2}[a_{11}\Delta x^2+2a_{12}\Delta x\Delta y+2a_{13}\Delta x\Delta z+a_{22}\Delta y^2+2a_{23}\Delta y\Delta z+a_{33}\Delta z^2]+\\&\frac{1}{2}[\alpha_{11}\Delta x^2+2\alpha_{12}\Delta x\Delta y+2\alpha_{13}\Delta x\Delta z+\alpha_{22}\Delta y^2+2\alpha_{23}\Delta y\Delta z+\alpha_{33}\Delta z^2]\end{aligned}\tag{1}$$

引入点 (x_0,y_0,z_0) 与 (x,y,z) 之间的距 $\rho=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$,从式(1)的括号内提出 ρ^2 并令 $\frac{\Delta x}{\rho}=\xi_1,\frac{\Delta y}{\rho}=\xi_2,\frac{\Delta z}{\rho}=\xi_3$,改写为 Δf 的表达式为

$$\Delta f=\frac{\rho^2}{2}[a_{11}\xi_1^2+2a_{12}\xi_1\xi_2+2a_{13}\xi_1\xi_3+a_{22}\xi_2^2+2a_{23}\xi_2\xi_3+a_{33}\xi_3^2]+\delta$$

其中 δ 是高阶无穷小.

由于 $\xi_1^2+\xi_2^2+\xi_3^2=1$,必定能找到这种正的常数 m,使得对于 ξ_1,ξ_2,ξ_3 可能有的一切数值总有 $a_{11}\xi_1^2+2a_{12}\xi_1\xi_2+2a_{13}\xi_1\xi_3+a_{22}\xi_2^2+2a_{23}\xi_2\xi_3+a_{33}\xi_3^2\geqslant m$,令

$$\begin{aligned}g(\xi_1,\xi_2,\xi_3)=&a_{11}\xi_1^2+2a_{12}\xi_1\xi_2+2a_{13}\xi_1\xi_3+a_{22}\xi_2^2+2a_{23}\xi_2\xi_3+a_{33}\xi_3^2=\\&(a_{12}^2\xi_1^2+2a_{12}\xi_1\xi_2+\xi_2^2)+(a_{13}^2\xi_1^2+2a_{13}\xi_1\xi_3+\xi_3^2)+(a_{23}^2\xi_2^2+2a_{23}\xi_2\xi_3+\xi_3^2)+\\&a_{11}\xi_1^2-a_{12}^2\xi_1^2-a_{13}^2\xi_1^2+a_{22}\xi_2^2-a_{23}^2\xi_2^2-\xi_2^2+a_{33}\xi_3^2-\xi_3^2-\xi_3^2=\\&(a_{12}\xi_1+\xi_2)^2+(a_{13}\xi_1+\xi_3)^2+(a_{23}\xi_2+\xi_3)^2+(a_{11}-a_{12}^2-a_{13}^2)\xi_1^2+\\&(a_{22}-a_{23}^2-1)\xi_2^2+(a_{33}-2)\xi_3^2\end{aligned}$$

所以,当 $\begin{cases}a_{11}-a_{12}^2-a_{13}^2>0\\a_{22}-a_{23}^2-1>0\\a_{33}-2>0\end{cases}$ 时,$f(x_0+\Delta x,y_0+\Delta y,z_0+\Delta z)-f(x_0,y_0,z_0)\geqslant 0$,$f(x,y,z)$ 在 (x_0,y_0,z_0) 处取极小值.

同理可证,当 $\begin{cases}a_{11}+a_{12}^2+a_{13}^2<0\\a_{22}+a_{23}^2+1<0\\a_{33}+2<0\end{cases}$ 时,$f(x,y,z)$ 在 (x_0,y_0,z_0) 处取极大值.

2 多元函数极值的判定方法及证明

在这一节中,对三元函数极值判定方法做一个推广,得到多元函数极值的判定方法.

定理 2 若函数 $f(x_1,x_2,x_3,\cdots,x_n)$ 存在驻点 $P_0=(x_1^0,x_2^0,\cdots,x_n^0)$,且在驻点处有二阶连续偏导数,令 $a_{11}=f_{x_1^2}(P_0)$,$a_{12}=f_{x_1x_2}(P_0)$,$\cdots$,$a_{1n}==f_{x_1x_n}(P_0)$,$a_{22}=f_{x_2^2}(P_0)$,$a_{23}=f_{x_2x_3}(P_0)$,$\cdots$,$a_{2n}=f_{x_2x_n}(P_0)$,$\cdots$,$a_{nn}=f_{x_nx_n}(P_0)$,则当 $\begin{cases}a_{11}-a_{12}^2-a_{13}^2-\cdots-a_{1n}^2>0\\a_{22}-a_{23}^2-a_{24}^2-\cdots-a_{2n}^2-1>0\\\vdots\\a_{nn}-(n-1)>0\end{cases}$ 时,$f(x_1,x_2,x_3,\cdots,x_n)$ 在 $(x_1^0,x_2^0,\cdots,x_n^0)$ 处取极小值;当 $\begin{cases}a_{11}+a_{12}^2+a_{13}^2+\cdots+a_{1n}^2>0\\a_{22}+a_{23}^2+a_{24}^2+\cdots+a_{2n}^2+1>0\\\vdots\\a_{nn}+(n-1)>0\end{cases}$ 时,$f(x_1,x_2,x_3,\cdots,x_n)$ 在 $(x_1^0,x_2^0,\cdots,x_n^0)$ 处取极大值.

证明　由 $f(P)$ 在点 P_0 处的泰勒公式:

$$f(P)=f(P_0)+\frac{\partial f(P_0)}{\partial x_1}(x_1-x_1^0)+\frac{\partial f(P_0)}{\partial x_2}(x_2-x_2^0)+\cdots+\frac{\partial f(P_0)}{\partial x_n}(x_n-x_n^0)+$$

$$\frac{1}{2}\left[\frac{\partial^2 f(P_0)}{\partial x_1^2}(x_1-x_1^0)^2+\frac{\partial^2 f(P_0)}{\partial x_1\partial x_2}(x_1-x_1^0)(x_2-x_2^0)+\cdots+\frac{\partial^2 f(P_0)}{\partial x_1\partial x_n}(x_1-x_1^0)(x_n-x_n^0)+\right.$$

$$\left.\cdots+\frac{\partial^2 f(P_0)}{\partial x_n\partial x_1}(x_n-x_n^0)(x_1-x_1^0)+\frac{\partial^2 f(P_0)}{\partial x_n\partial x_2}(x_n-x_n^0)(x_2-x_2^0)+\cdots+\frac{\partial^2 f(P_0)}{\partial x_n^2}(x_n-x_n^0)^2\right]+R_0=$$

$$f(P_0)+\operatorname{grad} f(P_0)^{\mathrm T}\begin{pmatrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_n\end{pmatrix}+\frac{1}{2}(\Delta x_1,\Delta x_2\cdots\Delta x_n)H_f(P_0)\begin{pmatrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_n\end{pmatrix}+R_n$$

其中 $\Delta x_i=x_i-x_i^0(i=1,2,\cdots,n)$,$R_n$ 是高阶无穷小.

对于驻点 P_0,$\operatorname{grad} f(P_0)=0$,则泰勒展开式又可写为

$$f(P)-f(P_0)=\frac{1}{2}(\Delta x_1,\Delta x_2,\cdots,\Delta x_n)H_f(P_n)(\Delta x_1,\Delta x_2\cdots\Delta x_n)^{\mathrm T}+R_n$$

由于 $f(x_1,x_2,x_3,\cdots,x_n)$ 的一切二阶偏导数在 $(x_1^0,x_2^0,\cdots,x_n^0)$ 连续,类似三元函数的证明过程,得到

$$\Delta f_1=\frac{1}{2}[a_{11}\Delta x_1^2+2a_{12}\Delta x_1\Delta x_2+\cdots+2a_{1n}\Delta x_1\Delta x_n+$$
$$a_{22}\Delta x_2^2+2a_{23}\Delta x_2\Delta x_3+\cdots+2a_{2n}\Delta x_2\Delta x_n+\cdots+a_{nn}\Delta x_n^2]\tag{2}$$

引入点 $(x_1^0,x_2^0,\cdots,x_n^0)$ 与 $(x_1,x_2,x_3,\cdots,x_n)$ 之间的距离 $\rho=\sqrt{\Delta x_1^2+\Delta x_2^2+\cdots+\Delta x_n^2}$,由式(2)的括号内提出 ρ^2,并令 $\frac{\Delta x_1}{\rho}=\xi_1,\frac{\Delta x_2}{\rho}=\xi_2,\cdots,\frac{\Delta x_n}{\rho}=\xi_n$,改写 Δf 的表达式为

$$\begin{aligned}\Delta f_1=&\frac{\rho^2}{2}[a_{11}\Delta\xi_1^2+2a_{12}\Delta\xi_1\Delta\xi_2+\cdots+2a_{1n}\Delta\xi_1\Delta\xi_n+\\&a_{22}\Delta\xi_2^2+2a_{23}\Delta\xi_2\Delta\xi_3+\cdots+2a_{2n}\Delta\xi_2\Delta\xi_n+\cdots+a_{nn}\Delta\xi_n^2]=\\&(a_{12}^2\xi_1^2+\xi_2^2+2a_{12}\xi_1\xi_2)+\cdots+(a_{1n}^2\xi_1^2+\xi_n^2+2a_{1n}\xi_1\xi_n)+\\&(a_{23}^2\xi_2^2+\xi_3^2+2a_{23}\xi_2\xi_3)+\cdots+(a_{2n}^2\xi_2^2+\xi_n^2+2a_{2n}\xi_2\xi_n)+\cdots\\&+(a_{n-1n}^2\xi_{n-1}^2+\xi_n^2+2a_{n-1n}\xi_{n-1}\xi_n)+a_{11}\xi_1^2-a_{12}^2\xi_1^2-a_{13}^2\xi_1^2-\cdots\\&-a_{1n}^2\xi_1^2+a_{22}\xi_2^2-a_{23}^2\xi_2^2-a_{24}^2\xi_2^2-\cdots-a_{2n}^2\xi_2^2-\xi_2^2+a_{33}\xi_3^2-a_{34}^2\xi_3^2-\\&\cdots-a_{3n}^2\xi_3^2-\xi_3^2-\xi_3^2+\cdots+a_{nn}\xi_n^2-(n-1)\xi_n^2\end{aligned}$$

所以,当 $\begin{cases}a_{11}-a_{12}^2-a_{13}^2-\cdots-a_{1n}^2>0\\a_{22}-a_{23}^2-a_{24}^2-\cdots-a_{2n}^2-1>0\\\vdots\\a_{nn}-(n-1)>0\end{cases}$ 时,$f(x_1,x_2,x_3,\cdots,x_n)$ 在 $(x_1^0,x_2^0,\cdots,x_n^0)$ 处取极小值.

同理,当 $\begin{cases}a_{11}+a_{12}^2+a_{13}^2+\cdots+a_{1n}^2>0\\a_{22}+a_{23}^2+a_{24}^2+\cdots+a_{2n}^2+1>0\\\vdots\\a_{nn}+(n-1)>0\end{cases}$ 时,$f(x_1,x_2,x_3,\cdots,x_n)$ 在 (x_0,y_0,z_0) 处取极大值.

3 实例应用

在这一节中,通过一个简单实例应用,对定理进行验证.

例 1 求函数 $f(x,y,z)=x^2+2y^2+3z^2+2x+4y-6z$ 的极值.

解 先求函数的驻点,由 $\begin{cases} f_x=2x+2=0 \\ f_y=4y+4=0 \\ f_z=6z-6=0 \end{cases}$ 得到函数的驻点为 $P_0(-1,-1,1)$,因为有 $a_{11}=2, a_{22}=4, a_{33}=6$, $a_{12}=0, a_{23}=0, a_{13}=0$,得到 $\begin{cases} a_{11}-a_{12}^2-a_{13}^2=4>0 \\ a_{22}-a_{23}^2-1=3>0 \\ a_{33}-2=4>0 \end{cases}$,所以 $P_0(-1,-1,1)$ 是函数的极小值点,在极值点处函数的极值为 $f(-1,-1,1)=-6$.

通过实例验证,不难发现这种新的判定方法能够判定函数的极值.

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The New Determination Method of Extremes of Multivariate Function

ZHAO Kun

(College of Mathematics and Systems Science,Shandong University of Science and Technology,
Shandong Qingdao 266590,China)

Abstract: The extremes of function have important research significance, and the solving method is varied. Based on the positive definiteness method of function with three variables, a new determination method about function with three variables and the proving process are got. This method also applies to conditions and unconditional extremes. Then this method is generalized to multivariate function, and a new determination method of extremes of multivariate function is got.

Key words: extremes;function with three variables;multivariate function; determination method