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M 矩阵与非负矩阵 Hadamard 积最小特征值的界*

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摘 要:给出 M 矩阵 A 的逆矩阵 A^{-1} 与 M 矩阵 B 的 Hadamard 积 $A\circ B^{-1}$ 的最小特征值 $q(A\circ B^{-1})$ 下界的新估计式;理论证明说明估计式提高了 Horn 在 1991 年给出的结果,数值算例说明估计式提高了一些现有的结果.

关键词:M 矩阵;非负矩阵;Hadamard 积;最小特征值;下界

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1 预备知识

令 $\mathbf{N}=\{1,2,\cdots,n\}$ 表示自然数集,$C^{n\times n}(\mathbf{R}^{n\times n})$ 表示 $n\times n$ 复(实)矩阵集,

下面给出将要用到的一些基础知识.

设 $A=(a_{ij})\in\mathbf{R}^{n\times n}$,若 $a_{ij}\geqslant 0$,则称 A 为非负矩阵($A\geqslant 0$);若 $a_{ij}\leqslant 0,i\neq j$,则称 A 为 Z 矩阵;进一步如果 A 为 Z 矩阵,且 $A^{-1}\geqslant 0$,就称 A 为非奇异 M 矩阵,并用 M_n 表示非奇异 M 矩阵的集合;若 A 是不可约非负矩阵,则存在正向量 u 使 $Au=\rho(A)u$,其中 u 称为 A 的右 Perron 特征向量;A 是不可约非奇异 M 矩阵,则存在正向量 v 使 $Av=T(A)v$,其中 v 称为 A 的右 Perron 特征向量.

设 $A=(a_{ij})\in\mathbf{R}^{n\times n}$,如果存在 $n\times n$ 置换矩阵 P,使 $P^{\mathrm{T}}AP=\begin{pmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{pmatrix}$,$A_{11}$ 是 $r\times r$ 子矩阵,A_{22} 是 $(n-r)\times(n-r)$ 子矩阵,$1\leqslant r<n$,则称 A 是可约的,否则 A 是不可约的.

矩阵 $A=(a_{ij}),B=(b_{ij})\in\mathbf{R}^{n\times n}$ 的 Hadamard 积为 $A\circ B=(a_{ij}b_{ij})\in\mathbf{R}^{n\times n}$.

令 $q(A)=\min\{\mathrm{Re}(\lambda):\lambda\in\sigma(A)\}$ 表示矩阵 A 的最小特征值,$\rho(A)$ 表示矩阵 A 的谱半径,$\sigma(A)$ 是 Z 矩阵 A 的特征值的集合.

若 $A,B\in M_n$,Fiedler M[1] 证明了 $A\circ B^{-1}\in M_n$.

2 主要结果

引理 1[2] 设 $A,B,C,D\in\mathbf{R}^{n\times n}$,其中 C,D 是对角矩阵,则

$$D(A\circ B)E=(DAE)\circ B=(DA)\circ BE=(AE)\circ(DB)=A\circ(DBE)$$

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引理 2[2] 设 $\boldsymbol{A}=(a_{ij})\in\mathbf{R}^{n\times n}$,则 $\boldsymbol{A}$ 的特征值位于下列集合

$$\bigcup_{i=1}^{n}\left\{z\in C:|z-a_{ii}|\leqslant\sum_{j\neq i}|a_{ij}|\right\}$$

定理 1 设 $\boldsymbol{A}=(a_{ij})\in\mathbf{R}^{n\times n}$,$\boldsymbol{B}=(b_{ij})\in\mathbf{R}^{n\times n}$是非奇异 $\boldsymbol{M}$ 矩阵,那么 $\boldsymbol{B}^{-1}=(\beta_{ij})$存在且 $\boldsymbol{B}^{-1}\geqslant 0$,则

$$q(\boldsymbol{A}\circ\boldsymbol{B}^{-1})\geqslant\max\{\min_{i}\{a_{ii}\beta_{ii}+\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})\},\min_{i}\{\beta_{ii}a_{ii}-\max_{j\neq i}|a_{ij}|(\rho(\boldsymbol{B}^{-1})-\beta_{ii})\}\}$$

证明

一方面设矩阵 $\boldsymbol{B}^{-1}\circ\boldsymbol{A}$ 不可约,那么 $\boldsymbol{A}$,$\boldsymbol{B}^{-1}$也不可约,令 $\boldsymbol{D}=\mathrm{diag}(d_1,d_2,\cdots,d_n)$,$d_i>0$,因为 $\boldsymbol{D}^{-1}\boldsymbol{B}^{-1}\boldsymbol{D}$ 为非负不可约矩阵,则存在正向量 $u=(u_1,u_2,\cdots,u_n)$,使得$(\boldsymbol{D}^{-1}\boldsymbol{B}^{-1}\boldsymbol{D})u=\rho(\boldsymbol{D}^{-1}\boldsymbol{B}^{-1}\boldsymbol{D})u=\rho(\boldsymbol{B}^{-1})u$,若写成分量形式,有 $\sum_{j\neq i}\frac{\beta_{ij}d_ju_j}{d_iu_i}=\rho(\boldsymbol{B}^{-1})-\beta_{ii}$.

再设 $\boldsymbol{U}=\mathrm{diag}(u_1,u_2,\cdots,u_n)$,令 $\boldsymbol{C}=(\boldsymbol{DU})^{-1}\boldsymbol{B}^{-1}(\boldsymbol{DU})$,则

$$\boldsymbol{C}=\begin{pmatrix}\beta_{11} & \frac{d_2u_2}{d_1u_1}\beta_{12} & \cdots & \frac{d_nu_n}{d_1u_1}\beta_{1n}\\ \frac{d_1u_1}{d_2u_2}\beta_{21} & \beta_{22} & \cdots & \frac{d_nu_n}{d_2u_2}\beta_{2n}\\ \vdots & \vdots & & \vdots\\ \frac{d_1u_1}{d_nu_n}\beta_{n1} & \frac{d_2u_2}{d_nu_n}\beta_{n2} & \cdots & \beta_{nn}\end{pmatrix}$$

这里的 $\boldsymbol{C}$ 为不约非负矩阵,

$$\boldsymbol{C}\circ\boldsymbol{A}=(s_{ij})\begin{pmatrix}\beta_{11}a_{11} & \frac{d_2u_2}{d_1u_1}\beta_{12}a_{12} & \cdots & \frac{d_nu_n}{d_1u_1}\beta_{1n}a_{1n}\\ \frac{d_1u_1}{d_2u_2}\beta_{21}a_{21} & \beta_{22}a_{22} & \cdots & \frac{d_nu_n}{d_2u_2}\beta_{2n}a_{2n}\\ \vdots & \vdots & & \vdots\\ \frac{d_1u_1}{d_nu_n}\beta_{n1}a_{n1} & \frac{d_2u_2}{d_nu_n}\beta_{n2}a_{n2} & \cdots & \beta_{nn}a_{nn}\end{pmatrix}$$

由引理 1 知$(\boldsymbol{DU})^{-1}(\boldsymbol{B}^{-1}\circ\boldsymbol{A})(\boldsymbol{DU})=(\boldsymbol{DU})^{-1}\boldsymbol{B}^{-1}(\boldsymbol{DU})\circ\boldsymbol{A}=\boldsymbol{C}\circ\boldsymbol{A}$,所以 $q(\boldsymbol{B}^{-1}\circ\boldsymbol{A})=q(\boldsymbol{C}\circ\boldsymbol{A})=\lambda$,又由引理 2 知存在 i,使得

$$|\lambda-\beta_{ii}a_{ii}|\leqslant\sum_{j\neq i}^{n}|s_{ij}|=\sum_{j\neq i}^{n}\left|\frac{d_ju_j\beta_{ij}a_{ij}}{d_iu_i}\right|\leqslant\max_{j\neq i}|a_{ij}|\sum_{j\neq i}^{n}\frac{d_ju_j\beta_{ij}}{d_iu_i}\leqslant\max_{j\neq i}|a_{ij}|(\rho(\boldsymbol{B}^{-1})-\beta_{ii})$$

即

$$\lambda\geqslant\beta_{ii}a_{ii}-\max_{j\neq i}|a_{ij}|(\rho(\boldsymbol{B}^{-1})-\beta_{ii})\geqslant\min_{i}\{\beta_{ii}a_{ii}-\max_{j\neq i}|a_{ij}|(\rho(\boldsymbol{B}^{-1})-\beta_{ii})\}$$

另一方面设矩阵 $\boldsymbol{A}\circ\boldsymbol{B}^{-1}$不可约,则 $\boldsymbol{A}$,$\boldsymbol{B}^{-1}$也不可约,令 $\boldsymbol{F}=\mathrm{diag}(f_1,f_2,\cdots,f_n)$,$f_i>0$,因为 $\boldsymbol{F}^{-1}\boldsymbol{AF}$ 为不可约非奇异 $\boldsymbol{M}$ 矩阵,则存在正向量 $v=(v_1,v_2,\cdots,v_n)$,使得$(\boldsymbol{F}^{-1}\boldsymbol{AF})v=q(\boldsymbol{F}^{-1}\boldsymbol{AF})v=q(\boldsymbol{A})v$,写成分量形式有 $\sum_{j\neq i}\frac{|a_{ij}|f_jv_j}{f_iv_i}=a_{ii}-q(\boldsymbol{A})$.

再设 $\boldsymbol{V}=\mathrm{diag}(v_1,v_2,\cdots,v_n)$,令 $\boldsymbol{G}=(\boldsymbol{FV})^{-1}\boldsymbol{A}(\boldsymbol{FV})$,则

$$\boldsymbol{G}=\begin{pmatrix} a_{11} & \frac{f_2v_2}{f_1v_1}a_{12} & \cdots & \frac{f_nv_n}{f_1v_1}a_{1n} \\ \frac{f_1v_1}{f_2v_2}a_{21} & a_{22} & \cdots & \frac{f_nv_n}{f_2v_2}a_{2n} \\ \vdots & \vdots & & \vdots \\ \frac{f_1v_1}{f_nv_n}a_{n1} & \frac{f_2v_2}{f_nv_n}a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$\boldsymbol{G}$ 为不可约非奇异 $\boldsymbol{M}$ 矩阵,

$$\boldsymbol{G}\circ\boldsymbol{B}^{-1}=(t_{ij})=\begin{pmatrix} a_{11}\beta_{11} & \frac{f_2v_2}{f_1v_1}a_{12}\beta_{12} & \cdots & \frac{f_nv_n}{f_1v_1}a_{1n}\beta_{1n} \\ \frac{f_1v_1}{f_2v_2}a_{21}\beta_{21} & a_{22}\beta_{22} & \cdots & \frac{f_nv_n}{f_2v_2}a_{2n}\beta_{2n} \\ \vdots & \vdots & & \vdots \\ \frac{f_1v_1}{f_nv_n}a_{n1}\beta_{n1} & \frac{f_2v_2}{f_nv_n}a_{n2}\beta_{n2} & \cdots & a_{nn}\beta_{nn} \end{pmatrix}$$

$\boldsymbol{G}\circ\boldsymbol{B}^{-1}$ 也为不可约非奇异 $\boldsymbol{M}$ 矩阵,又由引理 1 知,$(\boldsymbol{FV})^{-1}(\boldsymbol{A}\circ\boldsymbol{B}^{-1})(\boldsymbol{FV})=(\boldsymbol{FV})^{-1}\boldsymbol{A}(\boldsymbol{DU})\circ\boldsymbol{B}^{-1}=G\circ\boldsymbol{B}^{-1}$,即 $q(\boldsymbol{A}\circ\boldsymbol{B}^{-1})=q(G\circ\boldsymbol{B}^{-1})=\lambda$,又由引理 2 知存在 i,使得

$$|\lambda-a_{ii}\beta_{ii}|\leqslant\sum_{j=1,j\neq i}^{n}|t_{ij}|=\sum_{j=1,j\neq i}^{n}\left|\frac{f_jv_ja_{ij}\beta_{ij}}{f_iv_i}\right|\leqslant\max_{j\neq i}\beta_{ij}\sum_{j=1,j\neq i}^{n}\left|\frac{f_jv_ja_{ij}}{f_iv_i}\right|\leqslant-\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})$$

即

$$\lambda\geqslant\min_i\{a_{ii}\beta_{ii}+\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})\}$$

由 Hadamard 积的定义知 $\boldsymbol{A}\circ\boldsymbol{B}^{-1}=\boldsymbol{B}^{-1}\circ\boldsymbol{A}$,所以 $q(\boldsymbol{B}^{-1}\circ\boldsymbol{A})=q(\boldsymbol{A}\circ\boldsymbol{B}^{-1})$,即

$$q(\boldsymbol{A}\circ\boldsymbol{B}^{-1})\geqslant\max\{\min_i\{a_{ii}\beta_{ii}+\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})\},\min_i\{\beta_{ii}a_{ii}-\max_{j\neq i}|a_{ij}|(\rho(\boldsymbol{B}^{-1})-\beta_{ii})\}\}$$

对于可约的情况,类似于文献[3]中定理 2 的证明.

推论 1　设 $\boldsymbol{A}=(a_{ij})\in\mathbf{R}^{n\times n}$,$\boldsymbol{B}=(b_{ij})\in\mathbf{R}^{n\times n}$ 是非奇异 $\boldsymbol{M}$ 矩阵,$\boldsymbol{B}^{-1}=(\beta_{ij})$ 存在且 $\boldsymbol{B}^{-1}\geqslant0$,若 $\beta_{ii}\geqslant\max_{j\neq i}\beta_{ij}$,则

$$\max_i\{\min_i\{a_{ii}\beta_{ii}+\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})\},\min_i\{\beta_{ii}a_{ii}-\max_{j\neq i}|a_{ij}|(\rho(\boldsymbol{B}^{-1})-\beta_{ii})\}\}\geqslant q(\boldsymbol{A})\min_i\beta_{ii}$$

证明

$$\begin{aligned}&q(\boldsymbol{A})\min_i\beta_{ii}-a_{ii}\beta_{ii}-\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})\leqslant\\&q(\boldsymbol{A})\beta_{ii}-a_{ii}\beta_{ii}-\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})=\\&\beta_{ii}(q(\boldsymbol{A})-a_{ii})-\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})=\\&(q(\boldsymbol{A})-a_{ii})(\beta_{ii}-\max_{j\neq i}\beta_{ij})\end{aligned}$$

当 $\beta_{ii}\geqslant\max_{j\neq i}\beta_{ij}$ 时,$q(\boldsymbol{A})\min_i\beta_{ii}-a_{ii}\beta_{ii}-\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})\leqslant0$;

$$\begin{aligned}&q(\boldsymbol{A})\min_i\beta_{ii}-\beta_{ii}a_{ii}-\max_{j\neq i}|a_{ij}|(\rho(\boldsymbol{B}^{-1})-\beta_{ii})\leqslant\\&q(\boldsymbol{A})\beta_{ii}-\beta_{ii}a_{ii}+\max_{j\neq i}|a_{ij}|(\rho(\boldsymbol{B}^{-1})-\beta_{ii})\leqslant\\&\beta_{ii}(q(\boldsymbol{A})-a_{ii})+\max_{j\neq i}|a_{ij}|\rho(\boldsymbol{B}^{-1})-\max_{j\neq i}|a_{ij}|\beta_{ii}=\\&\beta_{ii}(q(\boldsymbol{A})-a_{ii}-\max_{j\neq i}|a_{ij}|)+\max_{j\neq i}|a_{ij}|\rho(\boldsymbol{B}^{-1})\leqslant\\&\rho(\boldsymbol{B}^{-1})(q(\boldsymbol{A})-a_{ii}-\max_{j\neq i}|a_{ij}|)+\max_{j\neq i}|a_{ij}|\rho(\boldsymbol{B}^{-1})=\\&\rho(\boldsymbol{B}^{-1})(q(\boldsymbol{A})-a_{ii}-\max_{j\neq i}|a_{ij}|+\max_{j\neq i}|a_{ij}|)^{-1}=\end{aligned}$$

$$\rho(\boldsymbol{B}^{-1})(q(\boldsymbol{A})-a_{ii})\leqslant 0$$

综合上面的分析知当 $\beta_{ii}\geqslant\max\limits_{j\neq i}\beta_{ij}$ 时,

$$\max_i\{\min_i\{a_{ii}\beta_{ii}+\max_{j\neq i}\beta_{ij}(q(\boldsymbol{A})-a_{ii})\},\min_i\{\beta_{ii}a_{ii}-\max_{j\neq i}|a_{ij}|(\rho(\boldsymbol{B}^{-1})-\beta_{ii})\}\}\geqslant q(\boldsymbol{A})\min_i\beta_{ii}$$

此处的估计式一定情况下提高了经典估计式 $q(\boldsymbol{A}\circ\boldsymbol{B}^{-1})\geqslant q(\boldsymbol{A})\min\limits_i\beta_{ii}$.

3 数值算例

设 $\boldsymbol{A}=\begin{pmatrix}4&-1&-1&-1\\-2&5&-1&-1\\0&-2&4&-1\\-1&-1&-1&4\end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix}1&-0.5&0&0\\-0.5&1&-0.5&0\\0&-0.5&1&-0.5\\0&0&-0.5&1\end{pmatrix}$, 应用 1991 年 Horn[4] 的结果:

$$q(\boldsymbol{A}\circ\boldsymbol{B}^{-1})\geqslant q(\boldsymbol{A})\min_i\beta_{ii}=1.6$$

应用 2004 年陈省生[5] 的结果:

$$q(\boldsymbol{A}\circ\boldsymbol{B}^{-1})\geqslant q(\boldsymbol{A})q(\boldsymbol{B})\min_i\left\{\left(\frac{a_{ii}}{q(\boldsymbol{A})}+\frac{b_{ii}}{q(\boldsymbol{B})}-1\right)\frac{\beta_{ii}}{b_{ii}}\right\}\geqslant 2.516\ 8$$

应用 2008 年黄荣[3] 的结果:

$$q(\boldsymbol{A}\circ\boldsymbol{B}^{-1})\geqslant\frac{1-\rho(\boldsymbol{J_A})\rho(\boldsymbol{J_B})}{1+\rho^2(\boldsymbol{J_B})}\min_i\left\{\frac{a_{ii}}{b_{ii}}\right\}=0.046\ 1$$

应用此处结果:

$$q(\boldsymbol{A}\circ\boldsymbol{B}^{-1})\geqslant 2.763\ 9$$

算例说明此处估计式提高了现有的结果.

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The Bound of Minimum Eigenvalue for the Hadamard Product of M Matrix and Nonnegative Matrix

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Abstract: The new lower bound of minimum eigenvalue $q(A\circ B^{-1})$ for the Hadamard product $A\circ B^{-1}$ of the inverse A^{-1} of MmatrixAand MmatrixB is given. It is prove that the estimator improves the results given by Horn in 1991, and numerical examples illustrate the estimator improves some existing results.

Key words: M matrix; nonnegative matrix; Hadamard product; minimum eigenvalue; lower bound