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一类推广的迭代泛函微分方程的光滑解

刘　佳
(山东城市建设职业学院 建筑经济管理系,济南 250103)

摘　要: 利用 Faà di Bruno 公式及 Schauder 不动点定理,证明了一类迭代泛函方程光滑解的存在性、唯一性和对给定函数的连续依赖性.

关键词: 迭代泛函微分方程;光滑解; Faà di Bruno 公式;不动点定理
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0　引　言

自 Jack Hale 的工作[1]发表后,关于泛函微分方程解的研究已有许多工作.形如

$$x'(t)=H(x^{(0)}(t),x^{(1)}(t),\cdots,x^{(m)}(t))$$

的迭代泛函微分方程,被许多人讨论过,这里 $x^{(0)}(t)=t,x^{(1)}(t)=x(t),\cdots,x^{(k)}(t)=x(x^{(k-1)}(t))$, $k=2,\cdots,m$. 确切地说,Eder[2]考虑了泛函微分方程 $x'(t)=x^{(2)}(t)$,证明了该方程的每一个解或者恒为零或者严格单调. 在文献[3, 4]中,Feckan 与王克在不同条件下研究了方程

$$x'(t)=f(x^{(2)}(t)) \tag{1}$$

此外,Stanek[5]考虑了方程 $x'(t)=x(t)+x^{(2)}(t)$, 得到了与文献[1]类似的结果.最近,司建国与其合作者[6, 7]讨论了方程 $x'(t)=x^{(m)}(t)$,$x'(t)=\dfrac{1}{x^{(m)}(t)}$和

$$x'(t)=\frac{1}{c_0x^{(0)}(t)+c_1x(t)+\cdots+c_mx^{(m)}(t)} \tag{2}$$

给出了解析解存在性的充分条件.特别是在文献[8]和[9]中,作者利用不动点定理,研究了方程 $x'(t)=\sum_{j=1}^{m}a_jx^{(j)}(t)+F(t)$,$x'(t)=\sum_{j=1}^{m}a_j(t)x^{(j)}(t)+F(t)$ 光滑解的存在性、唯一性及稳定性问题.在文献[10, 11]中, 赵侯宇分别讨论了式(2) 及一类推广的迭代泛函微分方程 $x'(t)=f(c_0t+c_1x(t)+\cdots+c_mx^{(m)}(t))$ 的光滑解的存在性.

此处利用不动点定理考虑一类更为广泛的迭代泛函微分方程

$$x'(t)=\frac{f(t)}{c_0t+c_1x(t)+\cdots+c_mx^{(m)}(t)} \tag{3}$$

光滑解的存在性.显然,当 $f(t)=1$ 时,式(2)是式(3)的特殊形式.那么,对于式(2)的光滑解的研究便可以看成是此处结论的特殊形式.

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作者简介: 刘佳(1983-),女,山东济南人,助教,从事基础数学研究.

可以证明方程(3)的局部光滑解的存在性连续依赖于光滑函数$f(t)$.类似于文献[7-11],先给出一些概念和记号.光滑函数是指一个函数有多次的连续导数且最高次的导数满足 Lipschitz 条件.若$x',\cdots,x^{(n)}$在区间I是连续的,记$x\in C^n(I,\mathbf{R})$.如果$x\in C^n(I,\mathbf{R})$且映闭区间I到I,记$x\in C^n(I,I)$.显然$C^n(I,\mathbf{R})$以范数$\|x\|_n=\sum_{k=0}^{n}\|x^{(k)}\|$,$\|x\|=\max_{t\in I}\{|x(t)|\}$构成 Banach 空间.对给定的常数$M_i>0(i=1,2,\cdots,n+1)$,记

$$\Omega(M_1,\cdots,M_{n+1};I)=\{x\in C^n(I,I):|x^{(i)}(t)|\leqslant M_i,i=1,2,\cdots,n$$
$$|x^{(n)}(t_1)-x^{(n)}(t_2)|\leqslant M_{n+1}|t_1-t_2|,t,t_1,t_2\in I\}$$

为了便于书写,记

$$x_{ij}(t)=x^{(i)}(x^{(j)}(t)),\quad x_{*jk}(t)=(x^{(j)}(t))^{(k)}$$

其中i,j,k是非负整数.为了寻找式(2)在$C^n(I,I)$中的解$x(t)$,使得$x(\xi)=\xi$,自然会想到在区间$[\xi-\delta,\xi+\delta]$中考虑,其中$\delta>0$.定义

$$\Psi(\xi;\eta_0,\cdots,\eta_{n-1};N_1,\cdots,N_n;I)=\{f\in\Omega(N_1,\cdots,N_n;I):f^{(i)}(\xi)=\eta_i,i=0,1,\cdots,n-1\}$$
$$X(\xi;\xi_0,\cdots,\xi_n;1,M_2,\cdots,M_{n+1};I)=\{x\in\Omega(1,M_2,\cdots,M_{n+1};I):x(\xi)=\xi_0,x^{(i)}(\xi)=\xi_i,i=1,2,\cdots,n\}$$

其中$\xi_0=\xi$.

由数学归纳法,对$k=0,1,\cdots,n$,可以证明

$$x_{*jk}(t)=P_{jk}(x_{10}(t),\cdots,x_{1,j-1}(t);\cdots;x_{k0}(t),\cdots,x_{k,j-1}(t))\tag{4}$$

$$\beta_{jk}(t)=P_{jk}(\overbrace{x'(\xi),\cdots,x'(\xi)}^{j};\cdots;\overbrace{x^{(k)}(\xi),\cdots,x^{(k)}(\xi)}^{j})\tag{5}$$

$$H_{jk}(t)=P_{jk}(\overbrace{1,\cdots,1}^{j};\overbrace{M_2,\cdots M_2}^{j};\cdots;\overbrace{M_k,\cdots M_k}^{j})\tag{6}$$

其中P_{jk}是系数为非负数的唯一多项式,式(4)-(6)的证明可在文献[8]中找到,I是$\mathbf{R}$上的闭区间.

1 主要定理

这一部分,证明方程(3)光滑解的存在性定理,需要用到下面的事实:对$x(t),y(t)\in X$,有

$$|x^{(j)}(t_1)-x^{(j)}(t_2)|\leqslant|t_1-t_2|,t_1,t_2\in I,j=0,1,\cdots,m\tag{7}$$

$$\|x^{(j)}-y^{(j)}\|\leqslant j\|x-y\|,j=1,2,\cdots,m\tag{8}$$

$$\|x-y\|\leqslant\delta^n\|x^{(n)}-y^{(n)}\|\tag{9}$$

不等式(7)-(9)的证明可在文献[9]中找到.

定理 1 设$I=[\xi-\delta,\xi+\delta]$,这里$\xi,\delta$满足

$$|c_0|-\sum_{i=0}^{m}|c_i|\geqslant 1,\xi\geqslant 0,\ 0<\delta\leqslant\frac{(|c_0|-\sum_{i=1}^{m}|c_i|)-1}{(|c_0|-\sum_{i=1}^{m}|c_i|)+1}\cdot\xi\tag{10}$$

且$f\in\Psi(\xi;\eta_0,\cdots,\eta_{n-1};N_1,\cdots,N_n;I)$,则式(3)在

$$X(\xi;\xi_0,\cdots,\xi_n;1,M_2,\cdots,M_{n+1};I)$$

中有解,其中

(i)

$$\xi_1=\eta_0\left(\xi\sum_{i=0}^{m}c_i\right)^{-1}\tag{11}$$

$$\xi_k=\sum_{j=0}^{k-1}\sum\frac{C_{k-1}^{j}(-1)^s s!\,(k-j-1)!\,\eta_j}{s_1!\,s_2!\cdots s_{k-j-1}!\,1!^{s_1}2!^{s_2}\cdots(k-j-1)!^{s_{k-j-1}}\left(\xi\sum_{i=0}^{m}c_i\right)^{s+1}}\left(\sum_{i=0}^{m}c_i\beta_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}c_i\beta_{ik-j-1}\right)^{s_{k-j-1}}\tag{12}$$

其中,$k=2,3,\cdots,n;s_1+2s_2+\cdots+(k-j-1)s_{k-j-1}=k-j-1;s=s_1+s_2+\cdots+s_{k-j-1}$.

(ii)

$$\sum_{j=0}^{k-1}\sum\frac{C_{k-1}^{j}s!\ (k-j-1)!\ N_j}{s_1!\ s_2!\ \cdots s_{k-j-1}!\ 1!^{s_1}2!^{s_2}\cdots(k-j-1)!^{s_{k-j-1}}\left(|c_0|-\sum\limits_{i=1}^{m}|c_i|\right)^{s+1}}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-j-1}\right)^{s_{k-j-1}}\leqslant M_k \tag{13}$$

其中,$k=2,3,\cdots,n;s_1+2s_2+\cdots+(k-j-1)s_{k-j-1}=k-j-1;s=s_1+s_2+\cdots+s_{k-j-1}$.

(iii)

$$\sum_{j=0}^{n-1}\sum\frac{C_{n-1}^{j}s!\ (n-j-1)!}{s_1!\ s_2!\ \cdots s_{n-j-1}!\ 1!^{s_1}2!^{s_2}\cdots(n-j-1)!^{s_{n-j-1}}}\Big\{N_{j+1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-j-1}\right)^{s_{n-j-1}}+(s+1)N_j(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-j-1}\right)^{s_{n-j-1}}+s_1N_j(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-j-1}\right)^{s_{n-j-1}}+\cdots+s_{n-j-1}N_j(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-j-1}\right)^{s_{n-j-1}-1}\left(\sum_{i=0}^{m}|c_i|H_{in-j}\right)\Big\}\leqslant M_{n+1} \tag{14}$$

其中,$s_1+2s_2+\cdots+(n-j-1)s_{n-j-1}=n-j-1,s=s_1+s_2+\cdots+s_{n-j-1}$.

证明　利用 Schauder 不动点定理来完成证明,定义算子

$$(Tx)(t)=\xi+\int_{\xi}^{t}f(t)\left(\sum_{i=0}^{m}c_ix^{(i)}(s)\right)^{-1}\mathrm{d}s \tag{15}$$

先证对 $\forall x\in X$,有 $Tx\in X$.由式(10)知

$$|(Tx)(t)-\xi|\leqslant\left|\int_{\xi}^{t}f(t)\left(\sum_{i=0}^{m}c_ix^{(i)}(s)\right)^{-1}\mathrm{d}s\right|\leqslant(\xi+\delta)(\xi-\delta)^{-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-1}|t-\xi|\leqslant\delta \tag{16}$$

因此,$(Tx)(I)\subseteq I$.由 Faà di Bruno 公式易知

$$(Tx)'(t)=f(t)\left(\sum_{i=0}^{m}c_ix^{(i)}(t)\right)^{-1} \tag{17}$$

$$(Tx)^{(k)}(t)=\sum_{j=0}^{k-1}C_{k-1}^{j}f^{(j)}(t)\left(\left(\sum_{i=0}^{m}c_ix^{(i)}(t)\right)^{-1}\right)^{(k-j-1)}=\sum_{j=0}^{k-1}\sum\frac{C_{k-1}^{j}(-1)^ss!\ (k-j-1)!\ f^{(j)}(t)}{s_1!\ s_2!\ \cdots s_{k-j}!\ 1!^{s_1}2!^{s_2}\cdots(k-j-1)!^{s_{k-j-1}}\left(\sum\limits_{i=0}^{m}c_ix^{(i)}(t)\right)^{s+1}}\left(\sum_{i=0}^{m}c_ix_{*i1}(t)\right)^{s_1}\cdots\left(\sum_{i=0}^{m}c_ix_{*ik-j-1}(t)\right)^{s_{k-j-1}} \tag{18}$$

其中,$k=2,3,\cdots,n;s_1+2s_2+\cdots+(k-j-1)s_{k-j-1}=k-j-1;s=s_1+s_2+\cdots+s_{k-j-1}$.再注意到$(Tx)(\xi)=\xi$,及式(11)(12),有

$$(Tx)'(\xi)=f(\xi)\left(\sum_{i=0}^{m}c_ix^{(i)}(\xi)\right)^{-1}=\eta_0\left(\xi\sum_{i=0}^{m}c_i\right)^{-1}=\xi_1 \tag{19}$$

$$(Tx)^{(k)}(\xi)=\sum_{j=0}^{k-1}\sum\frac{C_{k-1}^{j}(-1)^{s}s!\ (k-j-1)!\ \eta_j}{s_1!\ s_2!\ \cdots s_{k-j-1}!\ 1!^{s_1}2!^{s_2}\cdots(k-j-1)!^{s_{k-j-1}}\left(\xi\sum_{i=0}^{m}c_i\right)^{s+1}}\left(\sum_{i=0}^{m}c_i\beta_{i1}\right)^{s_1}\cdots$$

$$\left(\sum_{i=0}^{m}c_i\beta_{ik-j-1}\right)^{s_{k-j-1}}=\xi_k \tag{20}$$

其中,$k=2,3,\cdots,n;s_1+2s_2+\cdots+(k-j-1)s_{k-j-1}=k-j-1;s=s_1+s_2+\cdots+s_{k-j-1}$.

因此,$(Tx)^{(k)}(\xi)=\xi_k,k=0,1,\cdots,n$.又因为

$$|(Tx)'(t)|=\left|\left(\sum_{i=0}^{m}c_i x^{[i]}(t)\right)^{-1}\right||f(t)|\leqslant\frac{(\xi+\delta)}{(\xi-\delta)}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-1}<1=M_1 \tag{21}$$

由式(13)(14)有

$$|(Tx)^{(k)}(t)|\leqslant\sum_{j=0}^{k-1}\sum\frac{C_{k-1}^{j}s!\ (k-j-1)!\ N_j}{s_1!\ s_2!\ \cdots s_{k-j-1}!\ 1!^{s_1}2!^{s_2}\cdots(k-j-1)!^{s_{k-j-1}}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{s+1}}$$

$$\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-j-1}\right)^{s_{k-j-1}}\leqslant M_k \tag{22}$$

其中,$k=2,3,\cdots,n;s_1+2s_2+\cdots+(k-j-1)s_{k-j-1}=k-j-1;s=s_1+s_2+\cdots+s_{k-j-1}$.

$$|(Tx)^{(n)}(t_1)-(Tx)^{(n)}(t_2)|\leqslant\sum_{j=0}^{n-1}\sum\frac{C_{n-1}^{j}s!\ (n-j-1)!}{s_1!\ s_2!\ \cdots s_{n-j-1}!\ 1!^{s_1}2!^{s_2}\cdots(n-j-1)!^{s_{n-j-1}}}$$

$$\Big\{N_{j+1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots$$

$$\left(\sum_{i=0}^{m}|c_i|H_{in-j-1}\right)^{s_{n-j-1}}+(s+1)N_j(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-j-1}\right)^{s_{n-j-1}}+$$

$$s_1N_j(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-j-1}\right)^{s_{n-j-1}}+\cdots+$$

$$s_{n-j-1}N_j(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2}\cdots$$

$$\left(\sum_{i=0}^{m}|c_i|H_{in-j-1}\right)^{s_{n-j-1}-1}\left(\sum_{i=0}^{m}|c_i|H_{in-j}\right)\Big\}|t_1-t_2|\leqslant M_{n+1}|t_1-t_2 \tag{23}$$

到此,证明了 T 是一个将 X 映到自身的算子.

现在证明 T 的连续性.设 $x,y\in X$,则

$$\|Tx-Ty\|_n=$$

$$\|Tx-Ty\|+\|(Tx)'-(Ty)'\|+\sum_{k=2}^{n}\|(Tx)^{(k)}-(Ty)^{(k)}\|=$$

$$\max_{t\in I}\left|\int_{\xi}^{t}\left(f(s)\left(\left(\sum_{i=0}^{m}c_ix^{(i)}(s)\right)^{-1}-\left(\sum_{i=0}^{m}c_iy^{(i)}(s)\right)^{-1}\right)\right)\mathrm{d}s\right|+$$

$$\max_{t\in I}\left|f(t)\left(\left(\sum_{i=0}^{m}c_ix^{(i)}(t)\right)^{-1}-\left(\sum_{i=0}^{m}c_iy^{(i)}(t)\right)^{-1}\right)\right|+$$

$$\sum_{k=2}^{n}\sum_{j=0}^{k-1}\sum\frac{C_{k-1}^{j}s!\ (k-j-1)!\ |f^{(j)}(t)|}{s_1!\ s_2!\ \cdots s_{k-j}!\ 1!^{s_1}2!^{s_2}\cdots(k-j-1)!^{s_{k-j-1}}}$$

$$\Big\{\Big|\left(\sum_{i=0}^{m}c_ix^{(i)}(t)\right)^{-s-1}\left(\sum_{i=0}^{m}c_ix_{*i1}(t)\right)^{s_1}\cdots\left(\sum_{i=0}^{m}c_ix_{*ik-j-1}(t)\right)^{s_{k-j-1}}-$$

$$\left(\sum_{i=0}^{m}c_iy^{(i)}(t)\right)^{-s-1}\left(\sum_{i=0}^{m}c_iy_{*i1}(t)\right)^{s_1}\cdots\left(\sum_{i=0}^{m}c_iy_{*ik-j-1}(t)\right)^{s_{k-j-1}}\Big|\Big\}\leqslant$$

$$(\xi-\delta)^{-2}(\xi+\delta)\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\|x-y\|+$$
$$\sum_{k=2}^{n}\sum_{j=0}^{k-1}\sum\frac{C_{k-1}^{j}s!\,(k-j-1)!\,N_j}{s_1!\,s_2!\cdots s_{k-j}!\,1!^{s_1}2!^{s_2}\cdots(k-j-1)!^{s_{k-j-1}}}$$
$$\left\{(s+1)(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=0}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}c_i\|x^{(i)}(t)-y^{(i)}(t)\|\right)\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2}\cdots\right.$$
$$\left(\sum_{i=0}^{m}|c_i|H_{ik-j-1}\right)^{s_{k-j-1}}+s_1(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_1-1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\times\cdots\times$$
$$\left(\sum_{i=0}^{m}|c_i|H_{ik-j-1}\right)^{s_{k-j-1}}\left(\sum_{i=0}^{m}|c_i|\,\|x^{(i)}-y^{(i)}\|\right)+\cdots+s_{k-j-1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\times\cdots$$
$$\left.\times\left(\sum_{i=0}^{m}|c_i|H_{ik-j-2}\right)^{s_{k-j-2}}\left(\sum_{i=0}^{m}|c_i|H_{ik-j-1}\right)^{s_{k-j-1}-1}\left(\sum_{i=0}^{m}|c_i|H_{ik-j}\right)\left(\sum_{i=0}^{m}|c_i|\,\|x^{(i)}-y^{(i)}\|\right)\right\}+$$
$$\delta^{n+1}(\xi-\delta)^{-2}(\xi+\delta)\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\|x^{(n)}-y^{(n)}\| \tag{24}$$

经过计算,可以找到一列正数 P_k 使得

$$\sum_{k=2}^{n}\sum_{j=0}^{k-1}\sum\frac{C_{k-1}^{j}s!\,(k-j-1)!\,N_j}{s_1!\,s_2!\cdots s_{k-j}!\,1!^{s_1}2!^{s_2}\cdots(k-j-1)!^{s_{k-j-1}}}$$
$$\left\{(s+1)(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=0}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}c_i\|x^{(i)}(t)-y^{(i)}(t)\|\right)\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2}\cdots\right.$$
$$\left(\sum_{i=0}^{m}|c_i|H_{ik-j-1}\right)^{s_{k-j-1}}+s_1(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_1-1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\times\cdots\times$$
$$\left(\sum_{i=0}^{m}|c_i|H_{ik-j-1}\right)^{s_{k-j-1}}\left(\sum_{i=0}^{m}|c_i|\,\|x^{(i)}-y^{(i)}\|\right)+\cdots+s_{k-j-1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\times\cdots\times$$
$$\left.\left(\sum_{i=0}^{m}|c_i|H_{ik-j-2}\right)^{s_{k-j-2}}\left(\sum_{i=0}^{m}|c_i|H_{ik-j-1}\right)^{s_{k-j-1}-1}\left(\sum_{i=0}^{m}|c_i|H_{ik-j}\right)\left(\sum_{i=0}^{m}|c_i|\,\|x^{(i)}-y^{(i)}\|\right)\right\}\leqslant$$
$$\sum_{k=1}^{n-1}P_k(\xi,\delta,c_i,N_j,H_{ij})\|x^{(k)}-y^{(k)}\| \tag{25}$$

因此

$$\|Tx-Ty\|_n\leqslant$$
$$(\xi-\delta)^{-2}(\xi+\delta)\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\|x-y\|+$$
$$\sum_{k=1}^{n-1}P_k(\xi,\delta,c_i,N_j,H_{ij})\|x^{(k)}-y^{(k)}\|+$$
$$\delta^{n+1}(\xi-\delta)^{-2}(\xi+\delta)\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\|x^{(n)}-y^{(n)}\|\leqslant$$
$$\Gamma\|x-y\|_n \tag{26}$$

其中,$\Gamma=\max\left\{(\xi-\delta)^{-2}(\xi+\delta)\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right),\ \max_{1\leqslant k\leqslant n-1}\{P_k(\xi,\delta,c_i,N_j,H_{ij})\},\ \delta^{n+1}(\xi-\delta)^{-2}(\xi+\delta)\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\right\}$. 这里 $k=1,2,\cdots,n-1$;$s_1+2s_2+\cdots+(k-j-1)s_{k-j-1}=k-j-1$;$s=s_1+s_2+\cdots+s_{k-j-1}$.这就证明了 T 的连续性.类似文献[8-11],易知 X 是凸闭集,进一步还可证明 X 在 $C^n(I,I)$ 中一致有界,在 I 上等度连续,根据 Arzela-Ascoli 定理知 X 是 $C^n(I,I)$ 的相对紧子集.由 Schauder 不动点定理知,存在 $x(t)\in X$ 使得

$$x(t)=\xi+\int_{\xi}^{t}f(s)\left(\sum_{i=0}^{m}c_ix^{(i)}(s)\right)^{-1}ds \tag{27}$$

对式(27)两端求导即可看出 x 是式(3) 的解.定理证毕.

注意到,如果上面定理中有 $\Gamma<1$,则表明 T 是一个压缩算子.因此,上面证明中的不动点 x 必是唯一的.进一步可证这个唯一解关于给定的函数 f 是连续依赖的,即有定理2.

定理2 在定理1的条件下,且 $\Gamma<1$,则方程(3)在

$$X(\xi;\xi_0,\cdots,\xi_n;1,M_2,\cdots,M_{n+1};I)$$

中的唯一解连续依赖于给定的 f.

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Smoothing Solution to a Class of Generalized Iterative Functional Differential Equations

LIU Jia

(Department of Construction Economy Management, Shandong Urban Construction Vocational College, Jinan 250103, China)

Abstract: This paper uses Faà di Bruno formula and Schauder fixed point theory to prove the existence and uniqueness as well as continuous dependence for given functions of the smoothing solution to a class of iterative functional differential equations.

Key words: iterative functional differential equation; smoothing solution; Faà di Bruno formula; fixed point theory

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