文章编号:1672 - 058X(2012)10 - 0011 - 05

关于 Pell 方程 $ax^2 - mqy^2 = \pm 1$ ($m \in Z^+$, $2 \mid a$, $q \equiv \pm 1 \pmod 4$ 是素数)*

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摘 要:Pell 方程 $ax^2 - by^2 = \pm 1$($a,b \in Z^+$,a,b 不是完全平方数)可解性的判别是一个非常有意义的问题. 运用 Legendre 符号和同余的性质给出了形如 $ax^2 - mqy^2 = \pm 1$($m \in Z^+$,$2 \mid a$,$q \equiv \pm 1 \pmod 4$ 是素数,a,m,q 是非完全平方数)型 Pell 方程无正整数解的几个结论. 这些结论对研究狭义 Pell 方程 $x^2 - Dy^2 = \pm 1$(D 是非平方的正整数)起了重要作用.

关键词: Pell 方程;正整数解;素数;同余;Legendre 符号

中图分类号:O156. 1 **文献标志码**:A

关于 Pell 方程 $ax^2 - by^2 = 1$ 的整数解问题,文献[1]-[4]已有一些结果,此处旨在探讨 $ax^2 - mqy^2 = \pm 1$($m \in Z^+$,$q \equiv \pm 1 \pmod 4$ 是素数,a 为偶合数,a,m,q 是非完全平方数)型 Pell 方程的解的情况.

1 主要结论

定理 1 Pell 方程 $2\prod_{i=1}^{2s+1} p_i x^2 - mqy^2 = 1$($m \in Z^+$,$q \equiv \pm 1 \pmod 8$ 是素数,p_i 为奇素数,且 $\left(\frac{p_i}{q}\right) = -1$($i = 1,2,\cdots,2s+1$))无正整数解.

定理 2 Pell 方程 $2\prod_{i=1}^{s} p_i x^2 - mqy^2 = 1$($m \in Z^+$,$q \equiv \pm 3 \pmod 8$ 是素数,p_i 为奇素数,且 $\left(\frac{p_i}{q}\right) = 1$($i = 1,2,\cdots,s$))无正整数解.

定理 3 Pell 方程 $2\prod_{i=1}^{2s+1} p_i x^2 - mqy^2 = -1$($m \in Z^+$,$q \equiv 1,3 \pmod 8$ 是素数,p_i 为奇素数,且 $\left(\frac{p_i}{q}\right) = -1$($i = 1,2,\cdots,2s+1$))无正整数解.

定理 4 Pell 方程 $2\prod_{i=1}^{s} p_i x^2 - mqy^2 = -1$($m \in Z^+$,$q \equiv -1,-3 \pmod 8$ 是素数,p_i 为奇素数,且 $\left(\frac{p_i}{q}\right) = 1$($i = 1,2,\cdots,s$))无正整数解.

定理 5 Pell 方程 $2\prod_{i=1}^{2s+1} p_i \cdot \prod_{j=1}^{t} q_j x^2 - mqy^2 = 1$($m \in Z^+$,$q \equiv \pm 1 \pmod 8$ 是素数,p_i,q_j 为奇素数,且

收稿日期:2012 - 03 - 22;修回日期:2012 - 04 - 17.
* 基金项目:云南省教育厅科研基金(2011C121).
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$\left(\frac{p_i}{q}\right) = -1(i = 1,2,\cdots,2s+1)$,$\left(\frac{q_j}{q}\right) = 1(j = 1,2,\cdots,t)$)无正整数解.

定理 6 Pell 方程 $2\prod_{i=1}^{2s+1} p_i \cdot \prod_{j=1}^{t} q_j x^2 - mqy^2 = -1(m \in Z^+, q \equiv 1,3(\text{mod}8)$ 是素数,p_i, q_j 为奇素数,且 $\left(\frac{p_i}{q}\right) = -1(i = 1,2,\cdots,2s+1)$,$\left(\frac{q_j}{q}\right) = 1(j = 1,2,\cdots,t)$)无正整数解.

2 定理证明

2.1 定理 1 证明

证明 对 Pell 方程

$$2\prod_{i=1}^{2s+1} p_i x^2 - mqy^2 = 1 \tag{1}$$

两边取模 q 得:

$$2\prod_{i=1}^{2s+1} p_i x^2 \equiv 1(\text{mod}q) \tag{2}$$

若式(1)有正整数解,则式(2)有解,故有模 q 的 Legendre 符号值 $\left(\frac{2\prod_{i=1}^{2s+1} p_i}{q}\right) = 1$. 因 $q \equiv \pm 1(\text{mod}8)$,则$\left(\frac{2}{q}\right) = 1$. 又$\left(\frac{p_i}{q}\right) = -1(i = 1,2,\cdots,2s+1)$,则$\left(\frac{\prod_{i=1}^{2s+1} p_i}{q}\right) = -1$,故$\left(\frac{2\prod_{i=1}^{2s+1} p_i}{q}\right) = \left(\frac{2}{q}\right)\cdot\left(\frac{\prod_{i=1}^{2s+1} p_i}{q}\right) = -1$,矛盾.

所以式(1)无正整数解.

2.2 定理 2 证明

证明

$$2\prod_{i=1}^{s} p_i x^2 - mqy^2 = 1 \tag{3}$$

两边取模 q 得:

$$2\prod_{i=1}^{s} p_i x^2 \equiv 1(\text{mod}q) \tag{4}$$

若式(3)有正整数解,则式(4)有正整数解,故有模 q 的 Legendre 符号值 $\left(\frac{2\prod_{i=1}^{s} p_i}{q}\right) = 1$. 因 $q \equiv \pm 3(\text{mod}8)$,则$\left(\frac{2}{q}\right) = -1$. 又$\left(\frac{p_i}{q}\right) = 1(i = 1,2,\cdots,s)$,则$\left(\frac{\prod_{i=1}^{s} p_i}{q}\right) = 1$,故$\left(\frac{2\prod_{i=1}^{s} p_i}{q}\right) = \left(\frac{2}{q}\right)\cdot\left(\frac{\prod_{i=1}^{s} p_i}{q}\right) = -1$,矛盾.

所以式(3)无正整数解.

2.3 定理 3 证明

证明

$$2\prod_{i=1}^{2s+1} p_i x^2 - mqy^2 = -1 \tag{5}$$

两边取模 q 得:

$$2\prod_{i=1}^{2s+1} p_i x^2 \equiv -1(\mathrm{mod}q) \tag{6}$$

若式(5)有正整数解,则式(6)有正整数解,故有模 q 的 Legendre 符号值 $\left(\frac{-2\prod_{i=1}^{2s+1} p_i}{q}\right) = 1.$

当 $q \equiv 1(\mathrm{mod}8)$ 时, $\left(\frac{-1}{q}\right) = 1, \left(\frac{2}{q}\right) = 1$, 则 $\left(\frac{-2}{q}\right) = \left(\frac{-1}{q}\right) \cdot \left(\frac{2}{q}\right) = 1.$

当 $q \equiv 3(\mathrm{mod}8)$ 时, $\left(\frac{-1}{q}\right) = -1, \left(\frac{2}{q}\right) = -1$, 则 $\left(\frac{-2}{q}\right) = \left(\frac{-1}{q}\right) \cdot \left(\frac{2}{q}\right) = 1.$

又 $\left(\frac{p_i}{q}\right) = -1(i = 1,2,\cdots,2s+1)$, 则 $\left(\frac{\prod_{i=1}^{2s+1} p_i}{q}\right) = -1$, 故 $\left(\frac{-2\prod_{i=1}^{2s+1} p_i}{q}\right) = \left(\frac{-2}{q}\right) \cdot \left(\frac{\prod_{i=1}^{2s+1} p_i}{q}\right) = -1$, 矛盾.

所以式(5)无正整数解.

2.4　定理 4 证明

证明

$$2\prod_{i=1}^{s} p_i x^2 - mqy^2 = -1 \tag{7}$$

两边取模 q 得:

$$2\prod_{i=1}^{s} p_i x^2 \equiv -1(\mathrm{mod}q) \tag{8}$$

若式(7)有正整数解,则式(8)有解,故有模 q 的 Legendre 符号值 $\left(\frac{-2\prod_{i=1}^{s} p_i}{q}\right) = 1.$

当 $q \equiv -1(\mathrm{mod}8)$ 时, $\left(\frac{-1}{q}\right) = -1, \left(\frac{2}{q}\right) = 1$, 则 $\left(\frac{-2}{q}\right) = \left(\frac{-1}{q}\right) \cdot \left(\frac{2}{q}\right) = -1.$

当 $q \equiv -3(\mathrm{mod}8)$ 时, $\left(\frac{-1}{q}\right) = 1, \left(\frac{2}{q}\right) = -1$, 则 $\left(\frac{-2}{q}\right) = \left(\frac{-1}{q}\right) \cdot \left(\frac{2}{q}\right) = -1.$

又 $\left(\frac{p_i}{q}\right) = 1(i = 1,2,\cdots,s)$, 则 $\left(\frac{\prod_{i=1}^{s} p_i}{q}\right) = 1$, 故 $\left(\frac{-2\prod_{i=1}^{s} p_i}{q}\right) = = \left(\frac{-2}{q}\right)\left(\frac{\prod_{i=1}^{s} p_i}{q}\right) = -1$, 矛盾.

所以式(7)无正整数解.

2.5　定理 5 证明

证明　Pell 方程

$$2\prod_{i=1}^{2s+1} p_i \cdot \prod_{j=1}^{t} q_j x^2 - mqy^2 = 1 \tag{9}$$

两边取模 q 得:

$$2\prod_{i=1}^{2s+1} p_i \cdot \prod_{j=1}^{t} q_j x^2 \equiv 1(\mathrm{mod}q) \tag{10}$$

若式(9)有正整数解,则式(10)有解,故有模 q 的 Legendre 符号值 $\left(\frac{2\prod_{i=1}^{2s+1}p_i\cdot\prod_{j=1}^{t}q_j}{q}\right)=1$. 因 $q\equiv\pm1(\bmod 8)$,则 $\left(\frac{2}{q}\right)=1$. 又 $\left(\frac{p_i}{q}\right)=-1(i=1,2,\cdots,2s+1)$,$\left(\frac{p_j}{q}\right)=1(j=1,2,\cdots,t)$,则 $\left(\frac{\prod_{i=1}^{2s+1}p_i}{q}\right)=-1$,$\left(\frac{\prod_{i=1}^{t}p_j}{q}\right)=1$,故 $\left(\frac{2\prod_{i=1}^{2s+1}p_i\cdot\prod_{j=1}^{t}q_j}{q}\right)=\left(\frac{2}{q}\right)\cdot\left(\frac{\prod_{i=1}^{2s+1}p_i}{q}\right)\cdot\left(\frac{\prod_{i=1}^{t}p_j}{q}\right)=-1$,矛盾.

所以式(9)无正整数解.

2.6 定理 6 证明

证明 Pell 方程

$$2\prod_{i=1}^{2s+1}p_i\cdot\prod_{j=1}^{t}q_jx^2-mqy^2=-1 \tag{11}$$

两边取模 q 得:

$$2\prod_{i=1}^{2s+1}p_i\cdot\prod_{j=1}^{t}q_jx^2\equiv-1(\bmod q) \tag{12}$$

若式(11)有正整数解,则式(12)有解,故有模 q 的 Legendre 符号值 $\left(\frac{-2\prod_{i=1}^{2s+1}p_i\cdot\prod_{j=1}^{t}q_j}{q}\right)=1$.

当 $q\equiv1(\bmod 8)$ 时,$\left(\frac{-1}{q}\right)=1$,$\left(\frac{2}{q}\right)=1$,则 $\left(\frac{-2}{q}\right)=\left(\frac{-1}{q}\right)\cdot\left(\frac{2}{q}\right)=1$.

当 $q\equiv3(\bmod 8)$ 时,$\left(\frac{-1}{q}\right)=-1$,$\left(\frac{2}{q}\right)=-1$,则 $\left(\frac{-2}{q}\right)=\left(\frac{-1}{q}\right)\cdot\left(\frac{2}{q}\right)=1$.

又 $\left(\frac{p_i}{q}\right)=-1(i=1,2,\cdots,2s+1)$,$\left(\frac{p_j}{q}\right)=1(j=1,2,\cdots,t)$,则 $\left(\frac{\prod_{i=1}^{2s+1}p_i}{q}\right)=-1$,$\left(\frac{\prod_{i=1}^{t}p_j}{q}\right)=1$,故 $\left(\frac{-2\prod_{i=1}^{2s+1}p_i\cdot\prod_{j=1}^{t}q_j}{q}\right)=\left(\frac{-2}{q}\right)\cdot\left(\frac{\prod_{i=1}^{2s+1}p_i}{q}\right)\cdot\left(\frac{\prod_{i=1}^{t}p_j}{q}\right)=-1$,矛盾.

所以式(11)无正整数解.

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On Pell Equation $ax^2-mqy^2=\pm1$ ($m\in Z^+$, $2\mid a$, $q\equiv\pm1\ (mod\ 4)$, p *is a prime factor*)

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Abstract: The discrimination of solubility of Pell equation $ax^2-by^2=\pm1$ ($a,b\in Z^+$, ab is not a perfect square positive integer) is a very meaningful question. In this paper, by applying related knowledge of Legendre sign and nature of congruence, it works out several conclusions that Pell equation such as $ax^2-mqy^2=\pm1$ ($m\in Z^+$, $2\mid a$, $q\equiv\pm1(\bmod 4)$, p is a prime factor, a, m, q is not perfect square number) has not positive integer solution. These conclusions play an important role in studying restricted Pell equation $x^2-Dy^2=\pm1$ (D is a non-square positive integer).

Key words: Pell Equation; positive integer solution; prime factor; congruence; Legendre sign

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Remarks on Operator Coefficient in Taylor Formula for Vector Function of Hopf Bifurcation

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Abstract: A relatively perfect coefficient expression similar to a Hessian matrix in Taylor expanded formula for vector function of Hopf bifurcation $f: R^n\times R\to R^n$, which enhance visual recognition to operator coefficient of Taylor formula of vector function, here vector function $f(x_1,x_2,\cdots,x_n,\partial)=(f_1(x_1,x_2,\cdots,x_n,\partial), f_2(x_1,x_2,\cdots,x_n,\partial),\cdots,f_n(x_1,x_2,\cdots,x_n,\partial))^T$.

Key words: Hopf bifurcation; Taylor formula; vector function; operator

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