文章编号:1672 - 058X(2012)08 - 0031 - 05

正负相间双曲函数方幂和*

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摘　要:设 shx,chx 是双曲正,余弦函数,用发生函数的方法得到正负相间双曲正,余弦函数方幂与等比序列乘积之和:$\sum\limits_{k=0}^{n}(-1)^k d^k sh^r kx$,$\sum\limits_{k=0}^{n}(-1)^k d^k ch^r kx$ 和正负相间双曲正,余弦函数方幂与三角函数乘积和 $\sum\limits_{k=0}^{n}(-1)^k sh^r kx\sin k\beta$,$\sum\limits_{k=0}^{n}(-1)^k sh^r kx\cos k\beta$ 计算公式.

关键词:双曲函数;三角函数;等比序列; 形式幂级数;发生函数.

中图分类号:O156.1　　**文献标志码:**A

熟知双曲正,余弦函数定义[1]:$\mathrm{shx}=\frac{1}{2}(e^x-e^{-x})$,$\mathrm{chx}=\frac{1}{2}(e^x+e^{-x})$,令 $a=e^x$,$b=e^{-x}$,显然 $a-b=2\mathrm{shx}$;$a+b=2\mathrm{chx}$;$a\cdot b=1$;双曲函数研究有很多文献且在科学工程有广泛应用[2-5]. 文献[6]给出双曲函数方幂和计算公式. 提出研究双曲函数的数论性质,用发生函数的方法给出含有等比数列正负相间双曲正弦,余弦方幂和. 进而给出含有三角函数的正负相间双曲正弦,余弦方幂和计算公式.

定理 1　设双曲正,余弦函数 shx,chx 和复数 $d\neq 0$,则双曲函数方幂与等比序列乘积之和

当 $r=2s$ 时:

1) $$\sum_{k=0}^{n}(-1)^k d^k sh^r kx=\frac{1}{2^{r-1}}\sum_{i=0}^{s-1}\frac{(-1)^i\binom{r}{i}}{d^2+2dch(r-2i)x+1}[d^{n+2}ch(r-2i)nx+d^{n+1}ch(r-2i)(n+1)x+(-1)^n dch(r-2i)x+(-1)^n]+\frac{\binom{r}{s}}{2^r}h(n) \tag{1}$$

2) $$\sum_{k=0}^{n}(-1)^k d^k ch^r kx=\frac{1}{2^{r-1}}\sum_{i=0}^{s-1}\frac{\binom{r}{i}}{d^2+2dch(r-2i)x+1}[d^{n+2}ch(r-2i)nx+d^{n+1}ch(r-2i)(n+1)x+(-1)^n dch(r-2i)x+(-1)^n]+\frac{1}{2^r}\binom{r}{s}h(n) \tag{2}$$

其中 $h(n)=\begin{cases}(d^{n+1}+1)/(d+1), n=2m\\(d^{n+1}-1)/(d+1), n=2m+1\end{cases}$

当 $r=2s+1$ 时:

3) $$\sum_{k=0}^{n}(-1)^k d^k sh^r kx=\frac{1}{(-1)^s 2^{r-1}}\sum_{i=0}^{s}\frac{(-1)^i\binom{r}{i}}{d^2+2dch(r-2i)x+1}[d^{n+2}sh(r-2i)nx+d^{n+1}sh(r-2i)(n+1)x+(-1)^n dsh(r-2i)x] \tag{3}$$

收稿日期:2012 - 12 - 10;**修回日期:**2012 - 12 - 30.

* **基金项目:**银川能源学院科研基金项目(2011-37-15).

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4) $\sum_{k=0}^{n}(-1)^k d^k ch^r kx = \frac{1}{2^{r-1}}\sum_{i=0}^{s}\frac{\binom{r}{i}}{d^2+2dch(r-2i)x+1}[d^{n+2}ch(r-2i)nx + d^{n+1}ch(r-2i)(n+1)x + (-1)^n dch(r-2i)x + (-1)^n]$. (4)

证明 有理分式化成部分分式有结论:$\frac{1}{(1-az)}\frac{1}{1+z} = \frac{1}{a+1}\left[\frac{a}{1-az}+\frac{1}{1+z}\right]$

当 $r=2s$ 时:序列 $\{(-1)^n d^n sh^r nx\}$ 前 n 项的和发生函

$$G(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n}(-1)^k d^k sh^r kx)z^n = \frac{1}{2^r}[\sum_{i=0}^{r}\frac{(-1)^i\binom{r}{i}}{(1-a^{(r-i)}b^i dz)}](\frac{1}{1+z}) \quad (*)$$

$$G(z) = \frac{1}{2^r}\sum_{i=0}^{r}\frac{(-1)^i\binom{r}{i}}{a^{r-i}b^i d+1}[\frac{a^{r-i}b^i d}{1-a^{r-i}b^i dz}+\frac{1}{1+z}] =$$

$$\frac{1}{2^r}\{\sum_{i=0}^{s-1}\frac{(-1)^i\binom{r}{i}}{a^{r-i}b^i d+1}[\frac{a^{r-i}b^i d}{1-a^{r-i}b^i dz}+\frac{1}{1+z}] +$$

$$\sum_{i=0}^{s-1}\frac{(-1)^{r-i}\binom{r}{r-i}}{a^i b^{r-i}d+1}[\frac{a^i b^{r-i}d}{1-a^i b^{r-i}dz}+\frac{1}{1+z}] + \frac{(-1)^s\binom{r}{s}}{a^s b^s d+1}[\frac{a^s b^s d}{1-a^s b^s dz}+\frac{1}{1+z}]\} =$$

$$\frac{1}{2^r}\sum_{n=0}^{\infty}\{[a^{in+i}b^{(r-i)n+r-i}d^{n+1}+(-1)^n]\frac{(-1)^i\binom{r}{i}}{a^{r-i}b^i d+1}[a^{(r-i)n+r-i}b^{in+i}d^{n+1}+(-1)^n] +$$

$$\sum_{i=0}^{s-1}\frac{(-1)^{r-i}\binom{r}{r-i}}{a^i b^{r-i}d+1}[a^{in+i}b^{(r-i)n+r-i}d^{n+1}+(-1)^n] + \frac{(-1)^s\binom{r}{s}}{a^s b^s d+1}[a^{sn+s}b^{sn+s}d^{n+1}+(-1)^n]\}z^n$$

注意到 r 是偶数,有 $(-1)^{r-i}\binom{r}{r-i} = (-1)^i\binom{r}{i}$,及 $a\cdot b=1$,$(a^{r-i}b^i d+1)(a^i b^{r-i}d+1) = d^2+2dch(r-2i)x+1$,及 $\frac{(-1)^s\binom{r}{s}}{a^s b^s d+1}[a^{sn+s}b^{sn+s}d^{n+1}+(-1)^n] = h(n)$

其中 $h(n) = \begin{cases}(d^{n+1}+1)/(d+1), n=2m\\ (d^{n+1}-1)/(d+1), n=2m+1\end{cases}$

$$G(z) = \frac{1}{2^r}\sum_{n=0}^{\infty}\sum_{i=0}^{s-1}\frac{(-1)^i\binom{r}{i}}{d^2+2dch(r-2i)x+1}\{[a^{(r-i)n+r-i}b^{in+i}d^{n+1}+(-1)^n](a^i b^{r-i}d+1) +$$

$$[a^{in+i}b^{(r-i)n+r-i}d^{n+1}+(-1)^n](a^{r-i}b^i d+1) + (-1)^s\binom{r}{s}h(n)\}z^n =$$

$$\frac{1}{2^r}\sum_{n=0}^{\infty}\{\sum_{i=0}^{s-1}\frac{(-1)^i\binom{r}{i}}{d^2+2dch(r-2i)x+1}[d^{n+2}(ab)^{r+in}(a^{(r-2i)n}+b^{(r-2i)n}) +$$

$$(ab)^{i+in}(a^{(r-2i)n+(r-2i)}+b^{(r-2i)n+(r-2i)}) + (-1)^n(ab)^i d(a^{(r-2i)}+b^{(r-2i)}) +$$

$$2(-1)^n] + (-1)^s\binom{r}{s}h(n)\}z^n =$$

$$\frac{1}{2^{r-1}}\sum_{n=0}^{\infty}\{\sum_{i=0}^{s-1}\frac{(-1)^i\binom{r}{i}}{d^2+2dch(r-2i)x+1}[d^{n+2}ch(r-2i)nx + d^{n+1}ch(r-2i)(n+1)x +$$

$$(-1)^n dch(r-2i)x + (-1)^n + \frac{\binom{r}{s}}{2^r}h(n)]z^n$$

化简整理,比较(*)两端 z^n 的系数,得到式(1).

序列 $\{(-1)^n d^n ch^r nx\}$ 前 n 项的和发生函数为:

$$D(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n}(-1)^k d^k ch^r kx)z^n = [\frac{1}{2^r}\sum_{i=0}^{r}\frac{\binom{r}{i}}{(1-a^{(r-i)}b^i dz)}](\frac{1}{1+z})$$

使用式(1)给出的方法得式(2).

当 $r=2s+1$ 时,类似式(1)、(2)方法得式(3)、(4). 定理 1 证毕.

在定理 1,中令 $d=1$,有正负相间双曲正,余弦方幂和.

推伦 1　当 $r=2s$ 时:

1) $\sum_{k=0}^{n}(-1)^k sh^r kx = \frac{1}{2^r}\sum_{i=0}^{s-1}\frac{(-1)^i\binom{r}{i}}{1+ch(r-2i)x}[ch(r-2i)nx+ch(r-2i)(n+1)x+(-1)^n ch(r-2i)x+(-1)^n]+\frac{(-1)^s\binom{r}{s}}{2^r}h(n).$ (5)

2) $\sum_{k=0}^{n}(-1)^n ch^r kx = \frac{1}{2^r}\sum_{i=0}^{s-1}\frac{\binom{r}{i}}{1+ch(r-2i)x}[ch(r-2i)nx+ch(r-2i)(n+1)x+(-1)^n ch(r-2i)x+(-1)^n]+\frac{\binom{r}{s}}{2^r}h(n).$ (6)

其中 $h(n)=\begin{cases}1, n=2m\\0, n=2m+1\end{cases}$

当 $r=2s+1$ 时:

3) $\sum_{k=0}^{n}(-1)^k sh^r kx = \frac{1}{2^r}\sum_{i=0}^{s}\frac{(-1)^i\binom{r}{i}}{1+ch(r-2i)x}[sh(r-2i)nx+sh(r-2i)(n+1)x+(-1)^n sh(r-2i)x].$ (7)

4) $\sum_{k=0}^{n}(-1)^n ch^r kx = \frac{1}{2^r}\sum_{i=0}^{s}\frac{\binom{r}{i}}{1+ch(r-2i)x}[ch(r-2i)nx+ch(r-2i)(n+1)x+(-1)^n ch(r-2i)x+(-1)^n].$ (8)

定理 2　正负相间双曲正,余弦函数方幂与三角函数序列乘积之和

当 $r=2s$ 时:

1) $\sum_{k=0}^{n}(-1)^k sh^r kx\cos k\beta = \frac{1}{2^r}\sum_{i=0}^{s-1}\frac{(-1)^i\binom{r}{i}}{\cos\beta+ch(r-2i)x}[ch(r-2i)nx\cos(n+1)\beta+ch(r-2i)(n+1)x\cos n\beta+(-1)^n ch(r-2i)x+(-1)^n\cos\beta]+\frac{(-1)^s\binom{r}{s}}{2^r}p(n).$ (9)

2) $\sum_{k=0}^{n}(-1)^k sh^r kx\sin k\beta = \frac{1}{2^r}\sum_{i=0}^{s-1}\frac{(-1)^i\binom{r}{i}}{\cos\beta+ch(r-2i)x}[ch(r-2i)nx\sin(n+1)\beta+ch(r-2i)(n+1)x\sin n\beta-(-1)^n\sin\beta]+\frac{(-1)^s\binom{r}{s}}{2^r}q(n).$ (10)

3) $\sum_{k=0}^{n}(-1)^k ch^r kx\cos k\beta = \frac{1}{2^r}\sum_{i=0}^{s-1}\frac{\binom{r}{i}}{\cos\beta+ch(r-2i)x}[ch(r-2i)nx\cos(n+1)\beta+ch(r-2i)(n+1)x\cos n\beta+(-1)^n ch(r-2i)x+(-1)^n\cos\beta]+\frac{\binom{r}{s}}{2^r}p(n).$ (11)

4) $\sum_{k=0}^{n}(-1)^k ch^r kx\sin k\beta = \frac{1}{2^r}\sum_{i=0}^{s-1}\frac{\binom{r}{i}}{\cos\beta+ch(r-2i)x}[ch(r-2i)nx\sin(n+1)\beta+ch(r-2i)(n+1)x\sin n\beta-(-1)^n\sin\beta]+\frac{\binom{r}{s}}{2^r}q(n).$ (12)

其中 $p(n)=\begin{cases}\frac{1}{\cos(\beta/2)}\cos\frac{(n+1)\beta}{2}\cos\frac{n\beta}{2}, n=2m\\ \frac{-1}{\cos(\beta/2)}\sin\frac{(n+1)\beta}{2}\sin\frac{n\beta}{2}, n=2m+1\end{cases}$

$q(n)=\begin{cases}\frac{1}{\cos(\beta/2)}\cos\frac{(n+1)\beta}{2}\sin\frac{n\beta}{2}, n=2m\\ \frac{1}{\cos(\beta/2)}\sin\frac{(n+1)\beta}{2}\cos\frac{n\beta}{2}, n=2m+1\end{cases}$

当 $r=2s+1$ 时:

5) $\sum_{k=0}^{n}(-1)^k sh^r kx\cos k\beta = \frac{1}{(-1)^s 2^r}\sum_{i=0}^{s}\frac{(-1)^i\binom{r}{i}}{\cos\beta + ch(r-2i)x}[sh(r-2i)nx\cos(n+1)\beta + sh(r-2i)(n+1)x\cos n\beta + (-1)^n sh(r-2i)x + (-1)^n\cos\beta]$. (13)

6) $\sum_{k=0}^{n}(-1)^k sh^r kx\sin k\beta = \frac{1}{(-1)^s 2^r}\sum_{i=0}^{s}\frac{(-1)^i\binom{r}{i}}{\cos\beta + ch(r-2i)x}[sh(r-2i)nx\sin(n+1)\beta + sh(r-2i)(n+1)x\sin n\beta - (-1)^n\sin\beta]$. (14)

7) $\sum_{k=0}^{n}(-1)^k ch^r kx\cos k\beta = \frac{1}{2^r}\sum_{i=0}^{s}\frac{\binom{r}{i}}{\cos\beta + ch(r-2i)x}[ch(r-2i)nx\cos(n+1)\beta + ch(r-2i)(n+1)x\cos n\beta + (-1)^n ch(r-2i)x + (-1)^n\cos\beta]$. (15)

8) $\sum_{k=0}^{n}(-1)^k ch^r kx\sin k\beta = \frac{1}{2^r}\sum_{i=0}^{s}\frac{\binom{r}{i}}{\cos\beta + ch(r-2i)x}[ch(r-2i)nx\sin(n+1)\beta + ch(r-2i)(n+1)x\sin n\beta - (-1)^n\sin\beta]$. (16)

证明 令 $d=e^{j\beta}=\cos\beta+j\sin\beta; j=\sqrt{-1}$,

$$d^2+2dch(r-2i)x+1=(\cos2\beta+j\sin2\beta)+2(\cos\beta+j\sin\beta)ch(r-2i)x+1=$$
$$(\cos2\beta+2\cos\beta ch(r-2i)x+1)+j(\sin2\beta+2\sin\beta ch(r-2i)x)=$$
$$2\cos\beta(\cos\beta+ch(r-2i)x)+j[2\sin\beta(\cos\beta+ch(r-2i)x]=$$
$$2(\cos\beta+ch(r-2i)x)(\cos\beta+j\sin\beta)$$

所以 $\frac{1}{d^2+2dch(r-2i)x+1}=\frac{\cos\beta-j\sin\beta}{2(\cos\beta+ch(r-2i)x)}$. 另外当 $n=2m$ 时,计算

$$(d^{n+1}+1)/(d+1)=(\cos(n+1)\beta+j\sin(n+1)\beta+1)/(\cos\beta+j\sin\beta+1)=$$
$$2\cos\frac{(n+1)\beta}{2}(\cos\frac{(n+1)\beta}{2}+j\sin\frac{(n+1)\beta}{2})/[2\cos\frac{\beta}{2}(\cos\frac{\beta}{2}+j\sin\frac{\beta}{2})]=$$
$$\frac{1}{\cos(\beta/2)}\cos\frac{(n+1)\beta}{2}(\cos\frac{n\beta}{2}+j\sin\frac{n\beta}{2})$$

当 $n=2m+1$ 时,同法得 $(d^{n+1}-1)/(d+1)=\frac{1}{\cos(\beta/2)}\sin\frac{(n+1)\beta}{2}(-\sin\frac{n\beta}{2}+j\cos\frac{n\beta}{2})$.

$d^k=(\cos\beta+j\sin\beta)^k=\cos k\beta+j\sin k\beta$,将 d^k 代入式(1)左端

$$\sum_{k=0}^{n}(-1)^k d^k sh^r kx = \sum_{k=0}^{n} sh^r kx(\cos k\beta+j\sin k\beta) = \sum_{k=0}^{n} sh^r kx\cos k\beta + j\sum_{k=0}^{n} sh^r kx\sin k\beta$$

式(1)右端 $\frac{1}{2^{r-1}}\sum_{i=0}^{s-1}\frac{(-1)^i\binom{r}{i}}{2(\cos\beta+ch(r-2i)x)}(\cos\beta-j\sin\beta)\{[\cos(n+2)\beta ch(r-2i)nx+\cos(n+1)\beta ch(r-2i)(n+1)x+(-1)^n\cos\beta ch(r-2i)x+(-1)^n]+j[\sin(n+2)\beta ch(r-2i)nx+\sin(n+1)\beta ch(r-2i)(n+1)x+(-1)^n\sin\beta ch(r-2i)x]\}$.

两个复数相乘:

(1) 实部与实部相乘+虚部与虚部相乘,利用三角函数公式. $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$,整理化简,注意到 $(d^{n+1}-1)/(d-1)$ 整理复数形式将实部代入与式(1)左端实部相等得式(9).

(2) 前一复数实部乘以后一复数虚部+前一复数虚部乘以后一复数实部利用三角函数公式 $\sin(\alpha-\beta)=\sin\ \cos\beta-\cos\alpha\sin\beta$,整理化简注意到 $(d^{n+1}-1)/(d-1)$ 整理复数形式将虚部代入与式(1)左端虚部相等得式(10).

同法利用式(2)、(3)、(4)分别得式(11)、(12)、(13)、(14)和式(15)、(16).

定理2证毕.

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Power Sum Alternated with Positive and Negative of Hyperbolic Function

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Abstract: Let shx and chx be hyperbolic sine function and cosine function, the calculation formula $\sum_{k=0}^{n}(-1)^k d^k sh^r kx$ and $\sum_{k=0}^{n}(-1)^k d^k ch^r kx$, which are the product sum of the power alternated with positive and negative of hyperbolic sine function and cosine function and geometrical sequence, as well as the formula $\sum_{k=0}^{n}(-1)^k sh^r kx \sin k\beta$ and $\sum_{k=0}^{n}(-1)^k sh^r kx \cos k\beta$, which are the product sum of the power alternated with positive and negative of hyperbolic sine function and cosine function and trigonometric function, are obtained by function-generating method.

Key words: hyperbolic function; trigonometric function; geometrical sequence; formal power series; generating function

责任编辑：田　静