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一类具有分布时滞 Liénard 方程反周期解的存在性和唯一性*

田龙伟,王良龙,张洪彦
(安徽大学 数学科学学院,合肥 230039)

摘　要: 利用 Leray-Schauder 度理论,研究了一类具有分布时滞的 Liénard 方程反周期解的存在性和唯一性.

关键词: 分布时滞;Liénard 方程;Leray-Schauder 度理论;反周期解;存在唯一性
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在过去十几年中,人们对 Liénard 方程的周期解和概周期的存在性和唯一性进行了深入的研究[1-5]. 随着科学的发展和应用,对反周期解性质的研究逐渐引起人们的关注[6-12]. 在文献[7]中,作者利用 Leray-Schauder 度理论研究了一类 Liénard 方程 $x''+f(t,x'(t))+g(t,x(t-\tau(t)))=P(t)$ 的反周期解的存在性和唯一性. 在此基础上利用 Leray-Schauder 度理论讨论具有分布时滞的 Liénard 方程:

$$x''+f(t,x'(t))+g_1(t,x(t-\tau(t)))+h(t)\int_0^\infty K(s)g_2(x(t-s))\mathrm{d}s=P(t) \tag{1}$$

的反周期解的存在性和唯一性,推广了文献[7]中的结果.

其中, $g_2,h,\tau,e\in C(R\to R)$, $f,g_1\in C(R\times R\to R)$, $\int_0^\infty K(s)\mathrm{d}s<+\infty$ 和 $K(s)$ 是 $R^+=[0,+\infty)$ 上的连续函数,且满足:

(**H1**) 对任意的 $t,x\in\boldsymbol{R}$,都有 $f(t+\frac{T}{2},-x)=-f(t,x)$, $g_1(t+\frac{T}{2},-x)=-g_1(t,x)$, $g_2(-x)=g_2(x)$, $p(t+\frac{T}{2})=-p(t)$, $\tau(t+\frac{T}{2})=\tau(t)$, $h(t+\frac{T}{2})=-h(t)$.

定义 1　若 $u(t+\frac{T}{2})=-u(t)$,称连续函数 $u(t):R\to R$ 是 $\frac{T}{2}$ 反周期的. 显然当 $u(t+\frac{T}{2})=-u(t)$ 时, $u(t+T)=u(t)$,假设:

(**H2**) 存在一个非负常数 L_1 和 L_2,使得 $|g_1(t,x)-g_1(t,y)|\leqslant L_1|x-y|$, $|h(t)|\leqslant H$, $|g_2(x)-g_2(y)|=L_2|x-y|$, $\forall x,y\in\boldsymbol{R}$. 引入几个符号:

$C_T^k:=\{x\in C^k(R,R),x(t+T)=x(t),\forall t\in\boldsymbol{R}\}$, $k\in\{0,1,2,\cdots\}$

$C_T^{k,\frac{1}{2}}:=\{x\in C_T^k(R,R),x(t+\frac{T}{2})=-x(t),\forall t\in\boldsymbol{R}\}$

$|x|_q=(\int_0^T|x(t)|^q\mathrm{d}t)^{\frac{1}{q}}$, $|x|_\infty=\max\limits_{t\in[0,T]}|x(t)|$, $|x^{(k)}|_\infty=\max\limits_{t\in[0,T]}|x^{(k)}(t)|$

$\|x\|=\max\{|x|_\infty,|x'|_\infty,\cdots,|x|_\infty^{(k)}\}$, $\forall x\in C^{k,\frac{1}{2}}$

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作者简介: 田龙伟(1987-),男,河南三门峡人,硕士研究生,从事微分方程与动力系统研究.

1 准备知识

引理 1[13] 设 Ω 是线性赋范空间 X 中的有界开集，$\tilde{f}$ 是 $\overline{\Omega}$ 上的全连续场，如果 $\deg\{\tilde{f},\Omega,p\}\neq 0$，$p\in X\backslash f(\partial\Omega)$，则方程 $\tilde{f}(x)=p$ 在 Ω 内至少存在一个解.

引理 2[14] 设 $x\in C^2(R,R)$，且 $\forall t\in\boldsymbol{R}$，$x(t+T)=x(t)$，$\int_0^T x(t)\mathrm{d}t=0$，则

$$|x'(t)|_2^2\leqslant\left(\frac{T}{2\pi}\right)^2|x''(t)|_2^2 \tag{2}$$

$$|x|_\infty\leqslant\sqrt{\frac{T}{12}}|x'|_2 \tag{3}$$

引理 3 若方程(1)满足(H2)且满足下列条件之一：

(**H3**) 存在常数 L_3，使得

$$\frac{L_3T}{2\pi}+L_1\frac{\sqrt{3}T^2}{12}+HL_2\frac{\sqrt{3}T^2}{12}\int_0^\infty|K(s)|\mathrm{d}s<1$$

$$|f(t,x_1)-g_1(t,x_2)|\leqslant L_3|x_1-x_2|,\ \forall t,x_1,x_2\in\boldsymbol{R}$$

(**H4**) 存在常数 m，使得

$$f(t,u)=f(u),m|x_1-x_2|^2\leqslant(x_1-x_2)(f(x_1)-f(x_2)),\ \forall t,x_1,x_2\in\boldsymbol{R}$$

$$0\leqslant\frac{\sqrt{3}}{6}TL_1+\frac{\sqrt{3}}{6}THL_2\int_0^\infty|K(s)|\mathrm{d}s<m$$

则方程(1)至多存在一个 $\frac{T}{2}$ 解.

证明 假设 $x_1(t)$ 和 $x_2(t)$ 是方程(1)的两个 $\frac{T}{2}$ 反周期解，则

$$\begin{aligned}&(x_1-x_2)''+(f(t,x_1'(t))-f(t,x_2'(t)))+g_1(t,x_1(t-\tau(t)))-\\&g_1(t,x_2(t-\tau(t)))+h(t)\int_0^\infty K(s)(g_2(x_1(t-s))-g_2(x_2(t-s))\mathrm{d}s=0\end{aligned} \tag{4}$$

设 $z(t)=x_1(t)-x_2(t)$，从式(4)知道，

$$\begin{aligned}&(z(t))''+(f(t,x'_1(t))-f(t,x'_1(t)))+g_1(t,(x_1(t-\tau(t)))-\\&g_1(t,x_2(t-\tau(t)))+h(t)\int_0^\infty K(s)(g_2(x_1(t-s))-g_2(x_2(t-s))\mathrm{d}s=0\end{aligned} \tag{5}$$

因为 $z(t)=x_1(t)-x_2(t)$ 是定义在 R 上的反周期函数，则

$$\int_0^T z(t)\mathrm{d}t=\int_0^{\frac{T}{2}}z(t)\mathrm{d}t+\int_{\frac{T}{2}}^T z(t)\mathrm{d}t=\int_0^{\frac{T}{2}}z(t)\mathrm{d}t+\int_0^{\frac{T}{2}}z(t+\frac{T}{2})\mathrm{d}t=0 \tag{6}$$

由积分中值定理，存在 $\bar{\xi}\in[0,T]$，使得 $z(\bar{\xi})=0$.

由引理 3，有

$$|z_\infty|\leqslant\sqrt{\frac{T}{12}}|z'|_2 \tag{7}$$

假设(H3)或(H4)成立，有下列两种情况：

情况 1 如果(H3)成立，对方程(5)两边乘以 $-z(t)$ 且从 0 到 T 积分，有

$$\begin{aligned}&|z'|_2^2=-\int_0^T z''(t)z(t)\mathrm{d}t=\\&\int_0^T(f(x'_1(t))-f(x'_2(t)))z(t)\mathrm{d}t+\int_0^T(g(t,x_1(t-\tau(t)))-\end{aligned}$$

$$g(t,x_1(t-\tau(t))))z(t)\mathrm{d}t+\int_0^T h(t)\int_0^\infty K(s)(g_2(x_1(t-s))-$$

$$g_2(x_2(t-s)))z(t)\mathrm{d}s\mathrm{d}t\leqslant$$

$$L_3\int_0^T|x'_1(t)-x'_2(t)||z(t)|\mathrm{d}t+L_1\int_0^T|x_1(t-\tau(t))-x_2(t-\tau(t))||z(t)|\mathrm{d}t+$$

$$L_2\int_0^T\int_0^\infty|K(s)||x_1(t-s)-x_2(t-s)||h(t)||z(t)|\mathrm{d}s\mathrm{d}t=$$

$$L_3\int_0^T|z'(t)||z(t)|\mathrm{d}t+L_1\int_0^T|z(t-\tau(t))||z(t)|\mathrm{d}t+$$

$$L_2\int_0^T\int_0^\infty|K(s)||z(t-s)||h(t)||z(t)|\mathrm{d}s\mathrm{d}t$$

由式(2)(7)及 Schwarz 不等式,有

$$|z'|_2^2\leqslant L_3|z'(t)|_2|z(t)|_2+L_1|z|_\infty\int_0^T z(t)\mathrm{d}t+L_2H|z|_\infty\int_0^\infty K(s)\mathrm{d}s\int_0^T z(t)\mathrm{d}t\leqslant$$

$$L_3|z'(t)|_2|z(t)|_2+L_1|z|_\infty\sqrt{T}|z|_2+L_2H|z|_\infty\int_0^\infty K(s)\mathrm{d}s\sqrt{T}|z|_2\leqslant$$

$$\frac{L_3T}{2\pi}|z'|_2^2+L_1\sqrt{\frac{T}{12}}|z'|_2\sqrt{T}\frac{T}{2\pi}|z'|_2+L_2H\sqrt{\frac{T}{12}}|z'|_2\int_0^\infty K(s)\mathrm{d}s\sqrt{T}\frac{T}{2\pi}|z|_2\leqslant$$

$$(\frac{L_3T}{2\pi}+L_1\frac{\sqrt{3}T^2}{12\pi}+L_2H\frac{\sqrt{3}T^2}{12\pi}\int_0^\infty K(s)\mathrm{d}s|z'|_2^2 \tag{8}$$

因为 $z(t)$,$z'(t)$都是反周期连续函数,由条件(H3)和式(8),得 $z(t)\equiv z'(t)\equiv 0$, $\forall t\in\boldsymbol{R}$. 因此 $x_1(t)\equiv x_2(t)$, $\forall t\in\boldsymbol{R}$. 从而方程(1)至多有一个反周期解.

情况 2　如果(H4)成立,对方程(5)两边乘以 $z'(t)$且从 0 到 T 积分,有

$$m|z'|_2^2=\int_0^T m|x'_1(t)-x'_2(t)|^2\mathrm{d}t\leqslant$$

$$\int_0^T(f(x'_1)-f(x'_2))(x'_1(t)-x'_1(t))\mathrm{d}t=$$

$$-\int_0^T(g_1(t,x_1(t-\tau(t)))-g_1(t,x_2(t-\tau(t))))z'(t)\mathrm{d}t-$$

$$\int_0^T h(t)\int_0^\infty K(s)(g_2(t,x_1(t-s))-g_2(t,x_2(t-\tau s)))z'(t)\mathrm{d}s\mathrm{d}t\leqslant$$

$$L_1\int_0^T|x_1(t-\tau(t))-x_2(t-\tau(t))||z'(t)|\mathrm{d}t+$$

$$L_2H\int_0^T\int_0^\infty|K(s)||x_1(t-s)-x_2(t-s)||z'(t)|\mathrm{d}s\mathrm{d}t\leqslant$$

$$L_1|z|_\infty\int_0^T|z'(t)|\mathrm{d}t+L_2H|z|_\infty\int_0^\infty|K(s)|\mathrm{d}s\int_0^T|z'(t)|\mathrm{d}t\leqslant$$

$$\frac{\sqrt{3}}{6}TL_1|z'|_2^2+\frac{\sqrt{3}}{6}THL_2\int_0^\infty|K(s)|\mathrm{d}s|z'|_2^2=$$

$$(\frac{\sqrt{3}}{6}TL_1+\frac{\sqrt{3}}{6}THL_2\int_0^\infty|K(s)|\mathrm{d}s)|z'|_2^2 \tag{9}$$

由式(3)(9)和(H4),得到 $z(t)\equiv z'(t)\equiv 0$, $\forall t\in\boldsymbol{R}$. 因此 $x_1(t)\equiv x_2(t)$, $\forall t\in\boldsymbol{R}$. 从而方程(1)至多有一个反周期解.

2 主要结论及证明

定理1 设(H1)成立,如果(H3)和(H4)其中之一成立,则方程(1)有唯一的反周期解.

证明 构造方程(1)的辅助方程

$$x'' = -\lambda f(t,x'(t)) - \lambda g_1(t,x(t-\tau(t))) - \lambda h(t)\int_0^{\infty} K(s)g_2(x(t-s))\mathrm{d}s + \lambda P(s) = \lambda Q_1(t,x,x'),\lambda \in [0,1] \tag{10}$$

由引理3知,方程(1)至多有一个反周期解,因此要证明定理1,只要证明方程(1)至少有一个反周期解. 下面利用引理1来证明方程(1)至少有一个反周期解.

首先设 $x \in C_T^{1,\frac{1}{2}}$ 是辅助方程(10)的反周期解,类似(7)的证明过程,有

$$|Z|_{\infty} \leqslant \sqrt{\frac{T}{12}}|z'|_2 \tag{11}$$

对(H3)和(H4),考虑如下两种情况:

情况1 如果(H3)成立,对方程(10)两边乘以 $-x(t)$ 且从0到 T 积分,有

$$\begin{aligned}|x'|_2^2 = &-\int_0^T x''(t)x(t)\mathrm{d}t = \\
&\lambda\int_0^T f(t,x'(t))x(t)\mathrm{d}t + \lambda\int_0^T g_1(t,x(t-\tau(t)))x(t)\mathrm{d}t + \\
&\lambda\int_0^T h(t)\int_0^{\infty} K(s)g_2(x(t-s))x(t)\mathrm{d}s\mathrm{d}t - \lambda\int_0^T p(t)x(t)\mathrm{d}t \leqslant \\
&\int_0^T |f(t,x'_1(t)) - f(t,0)||x(t)|\mathrm{d}t + \int_0^T |f(t,0)||x(t)|\mathrm{d}t + \int_0^T | g_1(t,x(t-\tau(t))) - \\
&g_1(t,0)| |x(t)|\mathrm{d}t + \int_0^T |g_1(t,0)||x(t)|\mathrm{d}t + \int_0^T\int_0^{\infty} |h(t)||K(s)| | g_2(t,x(t-s) - \\
&g_2(0))| |x(t)|\mathrm{d}s\mathrm{d}t + \int_0^T\int_0^{\infty} |h(t)||K(s)||g_2(0)||x(t)|\mathrm{d}s\mathrm{d}t + \int_0^T |p(t)||x(t)|\mathrm{d}t \leqslant \\
&L_3|x'|_2|x|_2 + L_1|x|_{\infty}|x|_2\sqrt{T} + HL_2\sqrt{T}|x|_{\infty}|x|_2\int_0^{\infty}|K(s)|\mathrm{d}s + \\
&H\sqrt{T}|g_2(0)||x|_2\int_0^{\infty}|K(s)|\mathrm{d}s + [\max\{|f(t,0)| + \\
&|g(t,0)|:0 \leqslant t \leqslant T\} + |p|_{\infty}]|x|_2\sqrt{T} \leqslant \\
&L_3\frac{T}{2\pi}|x'|_2^2 + \frac{\sqrt{3}}{12\pi}L_1T^2|x'|_2^2 + \frac{\sqrt{3}}{12\pi}L_2HT^2\int_0^{\infty}|K(s)|\mathrm{d}s|x'|_2^2 + \\
&\frac{HT\sqrt{T}}{2\pi}|g_2(0)|\int_0^{\infty}|K(s)|\mathrm{d}s|x'|_2 + \\
&[\max\{|f(t,0)| + |g(t,0)|:0 \leqslant t \leqslant T\} + |p|_{\infty}]\frac{T\sqrt{T}}{2\pi}|x'|_2\end{aligned} \tag{12}$$

由(H3)知,存在一个常数 D_1 使得

$$|x'|_2 \leqslant D_1, |x|_{\infty} \leqslant D_1 \tag{13}$$

设 $t_1 \in [0,T]$,使得 $|x(t_1)| = \max_{t\in[0,T]}|x(t)|$,则 $x'(t_1) = 0$. 存在常数 D_2 满足式(14):

$$\begin{aligned}|x'(t)| = &|x'_1(t) + \int_{t_1}^{t} x''(s)\mathrm{d}s| \leqslant \\
&\int_{t_1}^{t} |f(s,x'(s)) + g_1(s,x(s-\tau(s))) + h(s)\int_0^{\infty}K(v)g_2(s,x(s-v))\mathrm{d}v - p(s)|\mathrm{d}s \leqslant\end{aligned}$$

$$\int_0^T |f(t,x'(t)-f(t,0))|\,\mathrm{d}t+\int_0^T |f(t,0)|\,\mathrm{d}t+\int_0^T |g_1(t,x(t-\tau(t)))-g_1(t,0)|\,\mathrm{d}t+$$
$$\int_0^T |g_1(t,0)|\,\mathrm{d}t+\int_0^T\int_0^\infty |h(t)|\,|K(s)|\,|g_2(t,x(t-s))-g_2(0)|\,\mathrm{d}s\mathrm{d}t+$$
$$\int_0^T\int_0^\infty |h(t)|\,|K(s)|\,|g_2(0)|\,\mathrm{d}s\mathrm{d}t\leqslant$$
$$L_3\sqrt{T}|x'|_2+L_1T|x|_\infty+L_2HT|x|_\infty\int_0^\infty |K(s)|\,\mathrm{d}s+HT|g_2(0)|\int_0^\infty |K(s)|\,\mathrm{d}s+$$
$$T[\max\{|f(t,0)-g(t,0)|:0\leqslant t\leqslant T\}+|p|_\infty]\leqslant D_2,\ \forall t\in[0,T] \tag{14}$$

情况 2　如果(H4)成立，对方程(10)两边乘以 $x'(t)$ 且从 0 到 T 积分，有

$$m|x'|_2^2=\int_0^T mx'(t)x'(t)\mathrm{d}t\leqslant$$
$$\int_0^T (f(x'(t))-f(0))x'(t)\mathrm{d}t=$$
$$-\int_0^T g_1(t,x(t-\tau(t)))x'(t)\mathrm{d}t-\int_0^T h(t)\int_0^\infty K(s)g_2(x(t-s))x'(t)\mathrm{d}s\mathrm{d}t+$$
$$\int_0^T p(t)x'(t)\mathrm{d}t-\int_0^T f(0)x'(t)\mathrm{d}t\leqslant$$
$$\int_0^T |g_1(t,x(t-\tau(t)))-g_1(t,0)|\,|x'(t)|\,\mathrm{d}t+$$
$$\int_0^T\int_0^\infty |h(t)|\,|K(s)|\,|g_2(x(t-s))-g_2(0)|\,|x'(t)|\,\mathrm{d}s\mathrm{d}t+$$
$$\int_0^T\int_0^\infty |h(t)|\,|K(s)|\,|g_2(0)|\,|x'(t)|\,\mathrm{d}s\mathrm{d}t+$$
$$\int_0^T |p(t)|\,|x'(t)|\,\mathrm{d}t+\int_0^T |f(0)|\,|x'(t)|\,\mathrm{d}t+\int_0^T |g_1(t,0)|\,|x'(t)|\,\mathrm{d}t\leqslant$$
$$L_1\int_0^T |x(t-\tau(t))|\,|x'(t)|\,\mathrm{d}t+HL_2\int_0^\infty |K(s)|\,\mathrm{d}s\int_0^T |x(t-s)|\,|x'(t)|\,\mathrm{d}t+$$
$$|g_2(0)|H\int_0^\infty |K(s)|\,\mathrm{d}s\int_0^T |x'(t)|\,\mathrm{d}t+\int_0^T[|p(t)|+|f(0)|+|g_1(t,0)|]\,|x'(t)|\,\mathrm{d}t\leqslant$$
$$\frac{\sqrt{3}}{6}L_1T|x'|_2^2+\frac{\sqrt{3}}{6}HL_2T\int_0^\infty |K(s)|\,\mathrm{d}s|x'|_2^2+$$
$$[\max\{|f(t,0)|+|g(t,0)|:0\leqslant t\leqslant T\}+|p|_\infty]\sqrt{T}|x'|_2 \tag{15}$$

因此存在常数 D_3，使得

$$|x'|_2\leqslant D_3\quad,\quad|x|_\infty\leqslant D_3 \tag{16}$$

成立.

同样对方程(10)两边乘以 $x''(t)$ 且从 0 到 T 积分，有

$$|x''|_2^2=\int_0^T |x''|^2\mathrm{d}t\leqslant$$
$$L_1\int_0^T |x(t-\tau(t))|\,|x''(t)|\,\mathrm{d}t+\int_0^T |g(t,0)|\,|x''(t)|\,\mathrm{d}t+$$
$$L_2H\int_0^T\int_0^\infty |K(s)|\,|x(t-s)|\,|x''(t)|\,\mathrm{d}s\mathrm{d}t+$$
$$H|g_2(0)|\int_0^T\int_0^\infty |K(s)|\,|x''(t)|\,\mathrm{d}s\mathrm{d}t+\int_0^T |p(t)|\,|x''(t)|\,\mathrm{d}t\leqslant$$
$$L_1D_3\sqrt{T}|x''|_2+L_2HD_3\sqrt{T}\int_0^\infty |K(s)|\,\mathrm{d}s|x''|_2+$$

$$H|g_2(0)|\sqrt{T}\int_0^{\infty}|K(s)|\mathrm{d}s|x''|_2+[\max_{t\in[0,T]}|g(t,0)|+|p(t)|_{\infty}]\sqrt{T}|x''|_2$$

由引理3知,存在一个常数 D_4,使得

$$|x'(t)|\leqslant\sqrt{\frac{\pi}{12}}|x''|_2\leqslant D_4,t\in[0,T] \tag{17}$$

因此对式(13)(14)(16)和(17),存在常数 $M_1>\max\{D_1+D_2,D_3+D_4\}$,使得 $\max\{|x|_{\infty},|x^1|_{\infty}\}<M_1$,设 $M=M_1+1$,使 $\Omega=\{x\in C_T^{1,\frac{1}{2}}=X:\max\{|x|_{\infty},|x^1|_{\infty}\}<M\}$.

显然当 $\lambda\in(0,1]$时,方程(10)在$\partial\Omega$没有反周期解. 下面证明方程(1)反周期解的存在性. 任取 $x\in C_T^{1,\frac{1}{2}}$,则 $x(t)$可以展成 Fourier 形式:

$$x(t)=\sum_{i=0}^{\infty}[a_{2i+1}\cos\frac{2\pi(2i+1)t}{T}+b_{2i+1}\sin\frac{2\pi(2i+1)t}{T}]$$

定义算子 $L:C_T^{1,\frac{1}{2}}\to C_T^{1,\frac{1}{2}}$ 如下:

$$(Lx)(t)=\int_0^T x(s)\mathrm{d}s-\frac{T}{2\pi}\sum_{i=0}^{\infty}\frac{b_{2i+1}}{2i+1}=\frac{T}{2\pi}\sum_{i=0}^{\infty}[\frac{a_{2i+1}}{2i+1}\sin\frac{2\pi(2i+1)t}{T}-\frac{b_{2i+1}}{2i+1}\cos\frac{2\pi(2i+1)t}{T}] \tag{18}$$

则$\frac{\mathrm{d}}{\mathrm{d}t}(Lx)(t)=x(t)$,且

$$|(Lx)(t)|\leqslant\int_0^T|x(t)|\mathrm{d}t+\frac{T}{2\pi}\sum_{i=0}^{\infty}\frac{b_{2i+1}}{2i+1}\leqslant T\|x\|+\frac{T}{2\pi}(\sum_{i=0}^{\infty}b_{2i+1}^2)^{\frac{1}{2}}(\sum_{i=0}^{\infty}\frac{1}{(2i+1)^2})^{\frac{1}{2}} \tag{19}$$

注意到

$$(\sum_{i=0}^{\infty}\frac{1}{(2i+1)^2})^{\frac{1}{2}}=\frac{\pi}{2\sqrt{2}}$$

由 Parseval 恒等式$\int_0^T|x(t)|^2\mathrm{d}t=\frac{T}{2}\sum_{i=0}^{\infty}[a_{2i+1}^2+b_{2i+1}^2]$,可以得到

$$|(Lx)(t)|\leqslant T\|x\|+\frac{T}{4\sqrt{2}}(\sum_{i=0}^{\infty}(a_{2i+1}^2+b_{2i+1}^2))^{\frac{1}{2}}\leqslant\|x\|+\frac{T}{4\sqrt{2}}(\frac{T}{2}\int_0^T|x(t)|^2\mathrm{d}t)^{\frac{1}{2}}\leqslant$$

$$(T+\frac{T}{4})\|x\|,t\in[0,T] \tag{20}$$

所以 $\|(Lx)(t)\|\leqslant(T+\frac{T}{4})\|x\|$,且算子 L 为连续算子.

对 $\forall x\in C_T^{1,\frac{1}{2}}$,由条件(H1)知 $Q_1(t+\frac{T}{2},x(t+\frac{T}{2}),x'(t+\frac{T}{2}))=-Q_1(t,x(t),x'(t))$,因此 $Q_1(t,x(t),x'(t))\in C_T^{0,\frac{1}{2}}$. 定义算子 $F_\mu:\overline{\Omega}\to C_T^{2,\frac{1}{2}}\subset X$ 如下:$F_\mu(x)=\lambda L(L(Q_1(x)))=\lambda L^2(Q_1(x)),\lambda\in[0,1]$,由 Arzela-Ascoli 引理易证 F_μ 为紧同伦,显然 F_1 在 $\overline{\Omega}$ 上的不动点即为方程(1)的反周期解.

定义同伦连续场:$H_\mu(x):\overline{\Omega}\times[0,1]\to C_T^{1,\frac{1}{2}}$,$H_\mu(x)=x-F_\mu(x)$,由 Ω 的定义知 $H_\mu(\partial\Omega)\neq0,\lambda\in[0,1]$,因此,由 Leray-Schauder 度的紧同论不变性知 $\deg\{x-F_1x,\Omega,0\}=\deg\{x,\Omega,0\}\neq0$.

由引理1知,方程 $x-F_1x=0$ 在 Ω 内至少有一个解,即算子 F_1 在 $\overline{\Omega}$ 上有唯一反周期解. 从而方程(1)有唯一的反周期解.

3 实 例

设$f(t,x)=\frac{1}{2\pi}x(t)|\sin x(t)|\cos^2 t,g(t,x)=\frac{1}{4}(1+\cos^4 t)\frac{1}{6\pi}\sin 2x(t)$,则 Liénard 方程:

$$x''(t)+\frac{1}{2\pi}x'(t)|\sin x'(t)|\cos^2(t)+\frac{1}{4}(1+\cos^4 t)\frac{1}{6\pi}\sin 2x(t-\cos(2t))+$$

$$\sin t\int_0^\infty e^{-2s}\frac{1}{3}\,|\,s(t-s)\,|\,ds = \frac{1}{2}\cos t \tag{21}$$

有唯一 π 反周期解.

证明　由式(21)知,$L_1=\frac{1}{2}$,$L_2=\frac{1}{3}$,$H=1$,$\int_0^\infty |e^{-2s}|ds=\frac{1}{2}$,$T=2\pi$ 和 $P(t)=\frac{1}{2}\cos t$.

容易验证方程(21)满足(H3)(H4),因此方程(21)存在唯一反周期解.

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Existence and Uniqueness of Anti-periodic Solutions to a Class of Liénard Equations with Continuously Distributed Delays

TIAN Long-wei,WANG Liang-long,ZHANG Hong-yan

(School of Mathematical Science,Anhui University,Hefei 230039,China)

Abstract: In this paper,we use the Leray-Schauder degree theory to study the existence and uniqueness of anti-periodic solution to a class of Liénard equations with continuously distributed delays.

Key words: continuously distributed delays; Liénard equation; Leray-Schauder degree theory; anti-periodic solution; existence and uniqueness

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