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# 无界域上二阶非线性脉冲边值问题解的存在性*

刘 锐,李树生
(安徽大学 数学科学学院,合肥 230039)

**摘 要**:利用压缩映像原理讨论了一类无界域上非线性二阶脉冲边值问题,得到了解存在的简单判别条件.

**关键词**:无界域;脉冲;边值问题;压缩映像原理
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## 1 简 介

近几年,由于在实际中的广泛应用,边值问题引起了许多学者的兴趣,得到了许多有意义的结果,如文献[1]中作者讨论了边值问题:$\begin{cases}x'(t)=f(t,x(t)),t>0\\ x(\infty)=\beta x(0)\end{cases}$.

具有脉冲影响的边值问题往往集中于周期边值问题中,但对于一般边界条件的脉冲边值问题,相应的结果还很少.基于上述原因,此处讨论如下边值问题:

$$(p(t)x'(t))'=f(t,x(t),x'(t)),t\in(0,+\infty)\backslash\{t_1,t_2,\cdots,t_n\} \tag{1}$$

边界条件为:

$$\begin{cases}\alpha_1 x(0)-\beta_1\lim\limits_{x\to 0^+}p(t)x'(t)=0\\ \alpha_2\lim\limits_{x\to+\infty}x(t)+\beta_2\lim\limits_{x\to+\infty}p(t)x'(t)=0\end{cases} \tag{2}$$

脉冲条件为:

$$\Delta x'(t_k)=I_k(x(t_k)) \tag{3}$$

其中$f\in C(R^+\times R^+\times R^+,R)$,$p\in C[0,+\infty)\cap C^1(0,+\infty)$且$p(t)>0$,$\forall t>0$,$\alpha_i,\beta_i\geqslant 0$,$i=1,2$,总假设$\rho=\alpha_1\beta_2+\alpha_2\beta_1+\alpha_1\alpha_2\int_0^{+\infty}\frac{\mathrm{d}\iota}{p(\iota)}\neq 0$,$I_k:R\to R$,$\Delta x'(t)=x'(t_k^+)-x'(t_k^-)$,$x'(t_k^+)$,$x'(t_k^-)$分别表示$x'(t)$在$t_k$处的左右极限,$k=1,2,\cdots,n$.

总假设:

(H1) 存在$L_1,L_2:[0,+\infty)\to[0,+\infty)$,且$\int_0^{+\infty}L_1(s)\mathrm{d}s<+\infty$,$\int_0^{+\infty}L_2(s)\mathrm{d}s<+\infty$,使得对于$\forall t\in[0,+\infty)$,$x_1,x_2,y_1,y_2\in R$,$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leqslant L_1(t)|x_1-x_2|+L_2(t)|y_1-y_2|$.

(H2) 存在常数$C_k>0$,$\forall x,y\in R$,使得$|I_k(x)-I_k(y)|\leqslant C_k|x_1-x_2|$,$k=1,2,\cdots,n$.

## 2 主要结果

设$PC=\{x\in C[0,+\infty):x(t)$有界,$t\in(0,+\infty)\}$,其范数为$|x|_0=\sup\limits_{x\in(0,+\infty)}|x(t)|$,定义$PC^1=\{x\in PC$:

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作者简介:刘锐(1985－),男,安徽合肥人,硕士研究生,从事微分方程与动力系统研究.

$x' \in C([0,+\infty)\backslash I)$有界,$x'(t_k^+)$,$x'(t_k^-)$存在且 $x'(t_k^-)=x'(t_k)\}$,其范数为 $\|x\|=\max\{|x|_0,|x'|_0\}$,则 $PC$,$PC^1$ 分别在各自范数下均为 Banach 空间,令 $I=\{t_1,t_2,\cdots,t_n\}$.

方程$(P(t)x'(t))=0$ 在式(2)下的解为:

$$G(t,s)=-\frac{1}{\rho}\begin{cases}(\beta_1+\alpha_1\int_0^s\frac{\mathrm{d}\iota}{p(\iota)})(\beta_2+\alpha_2\int_t^{+\infty}\frac{\mathrm{d}\iota}{p(\iota)}),s\leqslant t\\(\beta_1+\alpha_1\int_0^t\frac{\mathrm{d}\iota}{p(\iota)})(\beta_2+\alpha_2\int_s^{+\infty}\frac{\mathrm{d}\iota}{p(\iota)}),t\leqslant s\end{cases}$$

则 $G(t,s)$在$[0,+\infty)\times[0,+\infty)$上连续可微($t=s$ 时除外),且$\lim\limits_{t\to s^+}\frac{\partial}{\partial t}G(t,s)-\lim\limits_{t\to s^-}\frac{\partial}{\partial t}G(t,s)=\frac{1}{p(s)}$. 从而,对于 $i=1,2,\cdots,n$,有$\frac{\partial}{\partial t}G(t_i^-,t_k)=\frac{\partial}{\partial t}G(t_i^+,t_k)$,$k\neq i$;$p(t_k)\frac{\partial}{\partial t}G(t_i^+,t_k)-p(t_k)\frac{\partial}{\partial t}G(t_i^-,t_k)=1$.

根据 Green 函数的性质,可得:

**引理 1** 称函数 $x\in PC^1[0,+\infty)\cap C^2((0,+\infty)\backslash I)$是脉冲边值问题式(1)-(3)的解当且仅当 $x\in PC$ 满足:$x(t)=\int_0^{+\infty}G(t,s)f(s,x(s),x'(s))\mathrm{d}s+\sum\limits_{k=1}^{n}G(t,t_k)p(t_k)I_k(x(t_k))$,$t\in[0,+\infty)$.

给出下面记号:

$$D=\frac{1}{\rho}\sup_{t\in[0,+\infty)}\Big[\Big(|\beta_1|+|\alpha_1|\int_0^t\frac{\mathrm{d}\iota}{p(\iota)}\Big)\int_t^{+\infty}\Big(|\beta_2|+|\alpha_2|\int_s^{+\infty}\frac{\mathrm{d}\iota}{p(\iota)}\Big)(L_1(s)+L_2(s))\mathrm{d}s+\Big(|\beta_2|+|\alpha_2|\int_t^{+\infty}\frac{\mathrm{d}\iota}{p(\iota)}\Big)\int_0^t\Big(|\beta_2|+|\alpha_2|\int_0^s\frac{\mathrm{d}\iota}{p(\iota)}\Big)(L_1(s)+L_2(s))\mathrm{d}s\Big]$$

$$q=\sum_{k=1}^{n}\sup_{t\in[0,+\infty)}|G(t,t_k)|p(t_k)C_k$$

$$K_1=\frac{1}{\rho}\sup_{t\in[0,+\infty)}\Big[\Big(|\beta_1|+|\alpha_1|\int_0^t\frac{\mathrm{d}\iota}{p(\iota)}\Big)\int_t^{+\infty}\Big(|\beta_2|+|\alpha_2|\int_s^{+\infty}\frac{\mathrm{d}\iota}{p(\iota)}\Big)f(s,0,0)\mathrm{d}s+\Big(|\beta_2|+|\alpha_2|\int_t^{+\infty}\frac{\mathrm{d}\iota}{p(\iota)}\Big)\int_0^t\Big(|\beta_2|+|\alpha_2|\int_0^s\frac{\mathrm{d}\iota}{p(\iota)}\Big)f(s,0,0)\mathrm{d}s\Big]$$

$$K_2=\sum_{k=1}^{n}\sup_{t\in[0,+\infty)}|G(t,t_k)|p(t_k)|I_k(0)|$$

**定理 1** 在条件(H1)(H2)下,设 $K_1,K_2\geqslant0$,$D,q>0$,$D+q<1$,则对于任意的 $r>0$,当 $K_1+K_2+(D+q)r\leqslant r$ 时,脉冲边值问题式(1)-(3)有唯一解.

**证明** 设 $PC$ 中有界凸闭集 $S=\{x\in PC:\|x\|\leqslant r\}$. 定义映射 $T:S\to PC$,$(Tx)(t)=\int_0^{+\infty}G(t,s)f(s,x(s),x'(s))\mathrm{d}s+\sum\limits_{k=1}^{n}G(t,t_k)p(t_k)I_k(x(t_k))$,$t\in[0,+\infty)$. 显然,$Tx$ 连续,对于$\forall t\in[0,+\infty)$,有:

$$|(Tx)(t)|=\Big|\int_0^{+\infty}G(t,s)f(s,x(s),x'(s))\mathrm{d}s+\sum_{k=1}^{n}G(t,t_k)p(t_k)I_k(x(t_k))\Big|\leqslant$$

$$\sup_{t\in[0,+\infty)}\int_0^{+\infty}|G(t,s)||f(s,x(s),x'(s))|\mathrm{d}s+\sum_{k=1}^{n}|G(t,t_k)|p(t_k)|I_k(x(t_k))|\leqslant$$

$$\sup_{t\in[0,+\infty)}\int_0^{+\infty}|G(t,s)|(|f(s,x(s),x'(s))-f(s,0,0)|+|f(s,0,0)|)\mathrm{d}s+\sum_{k=1}^{n}|G(t,t_k)|p(t_k)(|I_k(x(t_k))-I_k(0)|+|I_k(0)|)\leqslant$$

$$\frac{1}{\rho}\sup_{t\in[0,+\infty)}\Big[\Big(|\beta_1|+|\alpha_1|\int_0^t\frac{\mathrm{d}\iota}{p(\iota)}\Big)\int_t^{+\infty}\Big(|\beta_2|+|\alpha_2|\int_s^{+\infty}\frac{\mathrm{d}\iota}{p(\iota)}\Big)(L_1(s)|x|_0+L_2(s)|x'|_0+|f(s,0,0)|)\mathrm{d}s+\Big(|\beta_2|+|\alpha_2|\int_t^{+\infty}\frac{\mathrm{d}\iota}{p(\iota)}\Big)\int_0^t\Big(|\beta_2|+|\alpha_2|\int_0^s\frac{\mathrm{d}\iota}{p(\iota)}\Big)(L_1(s)|x|_0+L_2(s)|x'|_0+|f(s,0,0)|)\mathrm{d}s\Big]+$$

$$\sup_{t\in[0,+\infty)}\left(\sum_{k=1}^{n}|G(t,t_k)|p(t_k)C_k|x|_0+\sum_{k=1}^{n}|G(t,t_k)|p(t_k)I_k\right)\leqslant$$
$$D\|x\|+K_1+K_2+q\|x\|=(D+q)\|x\|+K_1+K_2\leqslant(D+q)r+K_1+K_2\leqslant r$$

所以,$TS\subset S$,因为 $S$ 是 $PC$ 中的有界凸闭集,为了应用压缩映像原理,下面证明 $T$ 是一个压缩算子. 设 $x,u\in S,t\in[0,+\infty)$,有:

$$|(Tx)(t)-(Tu)(t)|\leqslant\int_0^{+\infty}|G(t,s)||f(s,x(s),x'(s))-f(s,u(s),u'(s)|\,ds+$$
$$\sum_{k=1}^{n}|G(t,t_k)|p(t_k)|I_k(x(t_k))-I_k(u(t_k))|\leqslant$$
$$\int_0^{+\infty}|G(t,s)|(L_1(s)|x(s)-u(s)|+L_2(s)|x'(s)-u'(s)|)ds+$$
$$\sum_{k=1}^{n}|G(t,t_k)|p(t_k)C_k|x(t_k)-u(t_k)|\leqslant$$
$$D\|x-u\|+q\|x-u\|\leqslant(D+q)\|x-u\|$$
i. e. $\|Tx-Tu\|\leqslant(D+q)\|x-u\|$

因为 $D+q<1$,所以 $T$ 是压缩的,由压缩映像原理可知:存在唯一的 $x\in S$ 使得 $Tx=x$. 证毕.

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# Existence of Second-order Nonlinear Boundary Value Problem with Impulsive Effects on an Unbounded Domain


**LIU Rui, LI Shu-sheng**

(School of Mathematical Sciences, Anhui University, Hefei 230039, China)



**Abstract**: By Banach contraction principle, we study a class of the second-order nonlinear boundary value problem with impulsive effects on an unbounded domain, simple determination conditions for the existence of solution are obtained.



**Key words**: unbounded domain; impulsive effects; boundary value problem; Banach contraction principle



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