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关于 B - 预不变凸函数的一个等价条件

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摘　要:在 $\eta(x, y)$满足条件 C, $f(x)$满足条件 D的情况下,给出了 B - 预不变凸函数的一个等价条件,通过上述等价条件将 R^n 上的实值函数转化为 $[0, 1]$上的实值函数,简化了问题.

关键词:不变凸集;B - 预不变凸函数;等价条件

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广义凸性函数在数学规划、最优化及最优控制等领域中具有十分重要的作用.因此,近年来对各种广义凸性的研究非常活跃. 1991年, Bector和 Singh在文献 [1]中引入了 B - 凸函数; 1993年, Bector, Singh和 Suneja在文献 [2]中引入了 B - 预不变凸函数.此处在他们研究的基础上给出了 B - 预不变凸函数的一个等价条件,将 R^n 上的实值函数转化为 $[0, 1]$上的实值函数.

1　基本概念

定义 1[3]　称集合 $X\subseteq R^n$ 是不变凸集,若 $\exists$向量值函数 $\eta: R^n \times R^n \to R^n$ 使得 $\forall x, y\in X, \lambda\in[0, 1], y+\lambda\eta(x, y)\in X$.

定义 2[2]　$X\subseteq R^n$ 是凸集,称 $f(x): X\to R$ 关于 b_1, b_2 是 B - 凸函数,若 $\forall x, y\in X, \lambda\in[0, 1]$,有 $f(y+\lambda(x-y))\le b_1(x, y,\lambda)f(y)+b_2(x, y,\lambda)f(x)$,其中 $b_1(x, y,\lambda)\ge 0, b_2(x, y,\lambda)\ge 0, b_1(x, y,\lambda)+b_2(x, y,\lambda)=1, b_1(x, y, 0)=1=b_2(x, y, 1)$.

定义 3[2]　$X\subseteq R^n$ 是不变凸集,称 $f(x): X\to R$ 关于 η, b_1, b_2 是 B - 预不变凸函数,若 $\exists$. $b: X\times X\times[0, 1]\to R^+, \eta: R^n\times R^n\to R^n$,使得 $\forall x, y\in X, \lambda\in[0, 1]$有 $f(y+\lambda\eta(x, y))\le b_1(x, y,\lambda)f(y)+b_2(x, y,\lambda)f(x)$. 其中, $b_1(x, y,\lambda)\ge 0, b_2(x, y,\lambda)\ge 0, b_1(x, y,\lambda)+b_2(x, y,\lambda)=1, b_1(x, y, 0)=1=b_2(x, y, 1)$.

条件 C[3]　$\eta: R^n\times R^n\to R^n$, $\eta(x, y)$满足条件 C是指 $\forall x, y\in R^n, \lambda\in[0, 1]$. $C_1: \eta(x, y+\lambda\eta(x, y))=-\lambda\eta(x, y)$; $C_2: \eta(x, y+\lambda\eta(x, y))=(1-\lambda)\eta(x, y)$. 显然由条件 C可知: $\forall x, y\in R^n, \forall\lambda\in[0, 1]$,

$$\begin{aligned}\eta(y+\lambda\eta(x, y), y) &= \eta(y+\lambda\eta(x, y), y+\lambda\eta(x, y)+\eta(y, y+\lambda\eta(x, y)))= \\ &-\eta(y, y+\lambda\eta(x, y))=\lambda\eta(x, y)\end{aligned}\tag{1}$$

条件 D[4]　$X\subseteq R^n$ 关于 $\eta(x, y)$为不变凸集,称 $f(x): X\to R$ 满足条件 D是指 $\forall x, y\in X, f(y+\eta(x, y))\le f(x)$.

2　结论及其证明

定理 1　$X\subseteq R^n$ 关于 $\eta(x, y)$为不变凸集, $f(x): X\to R$ 满足条件 D, $\eta(x, y)$满足条件 C,则 $f(x)$关于 η,

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b_1, b_2是 B - 预不变凸函数当且仅当 $\forall x, y\in X, \phi(\alpha) = f(y+\alpha\eta(x,y))$在 $[0,1]$上关于$\bar{b}_1, \bar{b}_2$是 B - 凸函数.

证明　必要性:

因为 $f(x)$在 X上关于 $b_1(x,y,\alpha), b_2(x,y,\alpha), \eta(x,y)$为 B - 预不变凸函数,由定义 $\forall x, y\in X, \alpha\in[0,1]$ s t $f(y+\alpha\eta(x,y)) \le b_1(x,y,\alpha)f(y) + b_2(x,y,\alpha))f(x)$. 其中 $b_1(x,y,\alpha)\ge 0, b_2(x,y,\alpha)\ge 0, b_1(x,y,\alpha) + b_2(x,y,\alpha) = 1, b_1(x,y,0) = 1 = b_2(x,y,1)$. 任取 $\alpha_1, \alpha_2\in[0,1], \lambda\in[0,1]$.

(a)当 $\alpha_1 = \alpha_2$时,

(i) 若 $\alpha_1 = \alpha_2 = 0, \phi(\alpha_2 + \lambda(\alpha_1 - \alpha_2)) = \phi(0) = f(y) = (1-\lambda)\phi(\alpha_2) + \lambda\phi(\alpha_1)$. 令 $\bar{b}_1(\alpha_1,\alpha_2,\lambda) = 1 - \lambda, \bar{b}_2(\alpha_1,\alpha_2,\lambda) = \lambda$,所以 $\phi(\alpha)$关于$\bar{b}_1, \bar{b}_2$为 B - 凸函数;

(ii) 若 $\alpha_1 = \alpha_2 = 1, \phi(\alpha_2 + \lambda(\alpha_1 - \alpha_2)) = \phi(1) = f(y+\eta(x,y)) = (1-\lambda)\phi(\alpha_2) + \lambda\phi(\alpha_1)$. 令 $\bar{b}_1(\alpha_1,\alpha_2,\lambda) = 1-\lambda, \bar{b}_2(\alpha_1,\alpha_2,\lambda) = \lambda$,所以 $\phi(\alpha)$关于$\bar{b}_1, \bar{b}_2$为 B - 凸函数;

(iii) 若 $\alpha_1, \alpha_2\in(0,1), \alpha_1 = \alpha_2, \phi(\alpha_2 + \lambda(\alpha_1 - \alpha_2)) = \phi(\alpha_2) = f(y+\alpha_2\eta(x,y)) = (1-\alpha_1\lambda)\phi(\alpha_2) + \alpha_1\lambda\phi(\alpha_1)$. 令 $\bar{b}_1(\alpha_1,\alpha_2,\lambda) = 1-\alpha_1\lambda, \bar{b}_2(\alpha_1,\alpha_2,\lambda) = \alpha_1\lambda$,所以 $\phi(\alpha)$关于$\bar{b}_1, \bar{b}_2$为 B - 凸函数.

(b) 当 $\alpha_1 > \alpha_2$时, $\alpha_1 - \alpha_2 > 0$,由 $\alpha_1, \alpha_2\in(0,1)$,可知 $\alpha_2\ne 1$,故 $0 < \dfrac{\alpha_1-\alpha_2}{1-\alpha_2} < 1$, 由条件 C和式 (1)可知:

$$\begin{aligned}
\eta(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y)) &= \eta(y+\alpha_2\eta(x,y) + (\alpha_1-\alpha_2)\eta(x,y), y+\alpha_2\eta(x,y)) = \\
&\eta(y+\alpha_2\eta(x,y) + \frac{\alpha_1-\alpha_2}{1-\alpha_2}\eta(x, y+\alpha_2\eta(x,y)), y+\alpha_2\eta(x,y)) = \\
&\frac{\alpha_1-\alpha_2}{1-\alpha_2}\eta(x, y+\alpha_2\eta(x,y)) = (\alpha_1-\alpha_2)\eta(x,y)
\end{aligned}$$

因 $f(x)$关于 η, b_1, b_2是 B - 预不变凸函数,故:

$$\begin{aligned}
\phi(\alpha_2+\lambda(\alpha_1-\alpha_2)) &= f(y+(\alpha_2+\lambda(\alpha_1-\alpha_2))\eta(x,y)) = \\
&f(y+\alpha_2\eta(x,y) + \lambda(\alpha_1-\alpha_2)\eta(x,y)) = \\
&f(y+\alpha_2\eta(x,y) + \lambda\eta(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y))) \le \\
&b_1(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y), \lambda)f(y+\alpha_2\eta(x,y)) + \\
&b_2(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y), \lambda)f(y+\alpha_1\eta(x,y))
\end{aligned}$$

令 $\bar{b}_1(\alpha_1,\alpha_2,\lambda) = b_1(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y), \lambda)$; $\bar{b}_2(\alpha_1,\alpha_2,\lambda) = b_2(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y), \lambda)$;则 $\phi(\alpha_2+\lambda(\alpha_1-\alpha_2)) \le \bar{b}_1(\alpha_1,\alpha_2,\lambda)\phi(\alpha_2) + \bar{b}_2(\alpha_1,\alpha_2,\lambda)\phi(\alpha_1)$. 故 $\phi(\alpha)$关于$\bar{b}_1, \bar{b}_2$是 B - 凸函数.

(c) 当 $\alpha_1 < \alpha_2$时, $\alpha_2 - \alpha_1 > 0$,由 $\alpha_1, \alpha_2\in(0,1)$,可知: $\alpha_1\ne 1$,故 $0 < \dfrac{\alpha_2-\alpha_1}{1-\alpha_1} < 1$. 由条件 C可知:

$$\begin{aligned}
\eta(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y)) &= \eta(y+\alpha_1\eta(x,y), y+\alpha_1\eta(x,y) + (\alpha_2-\alpha_1)\eta(x,y)) = \\
&\eta(y+\alpha_1\eta(x,y), y+\alpha_1\eta(x,y) + \frac{\alpha_2-\alpha_1}{1-\alpha_1}\eta(x, y+\alpha_1\eta(x,y))) = \\
&-\frac{\alpha_2-\alpha_1}{1-\alpha_1}\eta(x, y+\alpha_1\eta(x,y)) = (\alpha_1-\alpha_2)\eta(x,y)
\end{aligned}$$

因 $f(x)$关于 η, b_1, b_2是 B - 预不变凸函数,故:

$$\begin{aligned}
\phi(\alpha_2+\lambda(\alpha_1-\alpha_2)) &= f(y+(\alpha_2+\lambda(\alpha_1-\alpha_2))\eta(x,y)) = \\
&f(y+\alpha_2\eta(x,y) + \lambda(\alpha_1-\alpha_2)\eta(x,y)) = \\
&f(y+\alpha_2\eta(x,y) + \lambda\eta(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y))) \le \\
&b_1(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y), \lambda)f(y+\alpha_2\eta(x,y)) + \\
&b_2(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y), \lambda)f(y+\alpha_1\eta(x,y))
\end{aligned}$$

令 $\bar{b}_1(\alpha_1,\alpha_2,\lambda) = b_1(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y), \lambda)$; $\bar{b}_2(\alpha_1,\alpha_2,\lambda) = b_2(y+\alpha_1\eta(x,y), y+\alpha_2\eta(x,y),$

$\lambda)$;则 $\phi(\alpha_2+\lambda(\alpha_1-\alpha_2))\le \bar{b}_1(\alpha_1,\alpha_2,\lambda)\phi(\alpha_2)+\bar{b}_2(\alpha_1,\alpha_2,\lambda)\phi(\alpha_1)$. 所以 $\phi(\alpha)$是 B -凸函数.

充分性:因 $\phi(\alpha)$关于 $\bar{b}_1, \bar{b}_2$ 为 B -凸函数, $f(x)$满足条件 D,所以 $\forall\lambda\in[0,1]$, $\forall x,y\in X$,有:

$$
\begin{aligned}
f(y+\alpha\eta(x,y)) &= \phi(\alpha)=\phi(\alpha\cdot 1+(1-\alpha)\cdot 0)\le \\
&\bar{b}_1(1,0,\alpha)\phi(0)+\bar{b}_2(1,0,\alpha)\phi(1)= \\
&\bar{b}_1(1,0,\alpha)f(y)+\bar{b}_2(1,0,\alpha)f(y+\eta(x,y))\le \\
&\bar{b}_1(1,0,\alpha)f(y)+\bar{b}_2(1,0,\alpha)f(x)
\end{aligned}
$$

令 $b_1(x,y,\alpha)=\bar{b}_1(1,0,\alpha)$, $b_2(x,y,\alpha)=\bar{b}_2(1,0,\alpha)$,则 $f(y+\alpha\eta(x,y))\le b_1(x,y,\alpha)f(y)+b_2(x,y,\alpha)f(x)$.

综上所述, $f(x)$是 B -预不变凸函数当且仅当 $\forall x,y\in X$, $\phi(\alpha)=f(y+\alpha\eta(x,y))$在 $[0,1]$上是 B -凸函数.

推论 1 $X\subseteq R^n$ 为凸集, $f(x):X\to R$,则 $f(x)$关于 b_1, b_2 是 B -凸函数当且仅当 $\forall x,y\in X$, $\phi(\alpha)=f(y+\alpha(x-y))$在 $[0,1]$上关于 $\bar{b}_1, \bar{b}_2$ 是 B -凸函数.

证明 在定理 1中令 $\eta(x,y)=x-y$,显然 $\eta(x,y)$满足条件 C, $f(x)$满足条件 D.

3 结 论

此处定理 1利用条件 C和条件 D给出了 B -预不变凸函数的一个等价条件. 从定理和推论可看出,通过上述等价条件可以将 R^n 上的实值函数问题转化为 $[0,1]$上的实值函数问题,简化了问题.

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An equivalent condition of B-preinvex functions

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Abstract: In this paper, an equivalent condition of B-preinvex functions is obtained under the conditions which $\eta(x,y)$ satisfies condition C and $f(x)$ satisfies condition D, real value functions in R^n are turned into real value functions in [0,1] throngh above equivalent conditions, the problem is simplified.

Key words: invariant convexset; B-preinvex function; equivalent condition

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