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分数布朗运动下带有红利的最值期权定价*

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摘 要:依据投资者购买一家上市公司的股票,对公司进行投资,同时享受公司分红的权利,基于这样的实际情况,考虑在买卖最值期权支付红利的定价问题;假定股票的价格服从分数布朗运动驱动的随机微分方程,采用拉东-尼柯迪姆导数定理和多维哥萨诺夫定理,定义风险中性概率测度,同时建立风险中性概率测度下多维分数布朗运动每个股价的随机微分方程,在此基础上利用 Wick 积原理,求得每个股价的价格公式;运用风险中性定价的方法得到分数布朗运动下带有红利的最大值和最小值的看涨、看跌的期权定价公式以及平价公式;结果可在考虑股票支付红利的实际情况下,为研究最值期权定价问题提供理论参考。

关键词:分数布朗运动;支付红利;最值期权;风险中性定价

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0 引 言

期权作为一种金融衍生产品,其中期权定价理论是金融数学的核心内容之一。1973 年,诺贝尔经济学奖获得者布莱克(F. Black)和斯科尔斯(M. Scholes)[1]通过推导看涨期权定价公式,提出期权定价模型,随后同年默顿(R. Merton)[2]也发现同样的模型。正是布莱克-斯科尔斯期权定价模型和结果的出现,期权定价理论便得到迅速发展,同时为新兴衍生金融市场的各种以市价价格变动定价的衍生金融工具的合理定价奠定基础。资本市场的大量研究表明,经典布莱克-斯科尔斯期权定价模型过于理想化,金融资产(如股票)的对数收益率具有“尖峰尾厚”的特征,且价格变化也并非随机游走,而是呈现长期相关性和自相似性,因此布莱克-斯科尔斯期权定价模型不符合实际的金融市场。分数布朗运动具有长期依赖性、厚尾性和自相似性等特征,因此分数布朗运动成为研究标的资产价格过程一个有力工具[3]。1989 年 Peters[4]通过使用分数布朗运动来刻画资产价格变化规律。Duncan[5],Hu 和 Ksendal[6]应用 Wick 积和分数白噪声理论定义一种关于分数布朗运动的随机积分,即 wick-$It\hat{0}$ 型随机积分,并且证明由此定义的随机积分框架下的金融市场数学模型是无套利的且为完全市场。

传统的布莱克-斯科尔斯的期权定价模型假设基础资产不支付红利的情况下进行期权定价,这与实际情况不符合。在资本市场中,投资者购买一家上市公司股票,对公司进行投资,同时股东有权利要求公司支付红利,这就导致传统布莱克-斯科尔斯的期权定价模型与实际情况产生偏差。1985 年 McDonald[7]给出支付红利的期权定价公式,在此基础上文献[8]开始对美式、欧式支付红利的期权定价进行研究。

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为了刻画金融资产具有长记忆性和考虑实际期权标的资产股票支付红利的实际情况，在文献[9]的基础上引入红利率，利用风险中性原则和拟鞅方法给出分数布朗运动下带有红利的最值期权定价公式以及平价公式。

1 预备知识

分数布朗运动$\{B_t^H, t\geqslant 0\}$，是一个连续高斯过程，对所有的$t\geqslant 0$均值为零，且协方差函数$C_H(t,s)$满足：

$$C_H(t,s)=\frac{1}{2}\{|t|^{2H}+|s|^{2H}-|t-s|^{2H}\}$$

其中H称为分数布朗运动的 Hurst 指数或 Hurst 参数，且取值于$(0,1)$，当$H=\frac{1}{2}$时，称$B_H(t)$为标准的布朗运动。

分数布朗运动满足如下的性质：

平稳增量性：$B_t^H-B_S^H\overset{d}{=}B_{t-s}^H$, $\forall\, 0<s<t$；

自相似性：$|a|^{-H}B_{at}^H\overset{d}{=}B_t^H$, $\forall\, t>0$；$\mathrm{r}(n)=E(B_1^H(B_{n+1}^H-B_n^H))=\frac{1}{2}((n+1)^{2H}-2n^{2H}+(n-1)^{2H})$

因此，当$H>\frac{1}{2}$时，$\sum\limits_{n=0}^{\infty}r(n)=\infty$，称为长时记忆；而当$H<\frac{1}{2}$时，$\sum\limits_{n=0}^{\infty}r(n)<\infty$，称为短时记忆。

2 定价模型

为了得到分数布朗运动下带红利最值期权的定价模型，考虑到期日为时间T，敲定价格为K，并作假设：

1）市场是完备的，所有未定权益都是可复制的；

2）没有税收，没有交易费用，允许卖空；

3）不存在无风险套利机会；

4）交易是连续的；

在进行最值期权的定价，需要考虑涉及多个风险资产的价格演化模型。首先假设在金融市场中有种$n+1$证券，1 种无风险资产即债券，其价格满足方程：

$$\begin{cases}\mathrm{d}M_t=rM(t)\,\mathrm{d}t, 0\leqslant t\leqslant T\\ M(0)=1\end{cases}$$

n种风险资产如股票，设S_i是第i个风险资产价格$(i=1,\cdots,n)$，它适合随机微分方程：

$$\begin{cases}\mathrm{d}S_i(t)=(\mu_i-q_i)S_i(t)\,\mathrm{d}t+\\ \qquad S_i(t)\sum\limits_{j=1}^{n}\sigma_{ij}\mathrm{d}B_{H_i}(t), 0\leqslant t\leqslant T\\ S_i(0)>0\end{cases}$$

其中，r为无风险利率，μ_i为第i个风险资产预期收益率，q_i为第i个资产的红利率，$\sigma_{ij}(i,j=1,2,\cdots,n)$为波动率，并且$r,\mu_i,q_i,\sigma_{ij}(i,j=1,2,\cdots n)$都为常数，$\sum=(\sigma_{ij})_{n\times n}$为可逆矩阵，$B_{H_1}(t),B_{H_2}(t),\cdots,B_{H_n}(t)$为概率空间$(\Omega,F,P)$上的 Hurst 参数分别为$H_1,H_2,\cdots,H_n(0<H_i<1, i=1,2,\cdots,n)$的分数布朗运动，用$F_t$表示由分数布朗运动$B_{H_1}(t),B_{H_2}(t),\cdots,B_{H_n}(t)$产生的自然$\sigma$-流。

为了使用风险中性方法对最值期权进行定价，下面需要构造风险中性概率测度$\hat{P}$。由分数布朗运动的构造知，存在(Ω,F,P)上n维标准布朗运动$B_1(t),B_2(t),\cdots B_n(t)$，使得

$$B_{H_i}(t)=\int_R M_{H_i}(0,t)(s)\,\mathrm{d}B_i(s), i=1,2,\cdots,n$$

其中，$M_{H_i}(i=1,2,\cdots,n)$的定义可以参考文献[10]中式(2.28)。

方便起见设$\vec{\mu}=(\mu_1,\mu_2,\cdots,\mu_n)'$，$\vec{1}_n=(1,1,\cdots,1)'$，$\vec{B}(t)=(B_1(t),B_2(t),\cdots,B_n(t))'$，并令$\theta=\Sigma^{-1}(\vec{\mu}-r\vec{1}_n)$．即$\theta_i$满足

$$\sum_{j=1}^{n}\sigma_{ij}\theta_j=\mu_i-r, i=1,2,\cdots,n$$

根据拉东-尼柯迪姆导数定理，定义新的概率测度$\hat{P}$满足：

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}|F_T=\exp\left\{-\frac{1}{2}\vec{\theta}'\vec{\theta}T-\vec{\theta}'\vec{B}(T)\right\}$$

并令$\vec{\hat{B}}(t)=\vec{\theta}t+\vec{B}(t)$, $0\leqslant t\leqslant T$，则由 Girsanov 定理知，$\vec{\hat{B}}(t)=\{\hat{B}_1(t),\hat{B}_2(t),\cdots,\hat{B}_n(t)\}$是关于$\hat{P}$的$n$维标准布朗运动。令

$$\hat{B}_{H_1}(t)=\int_R M_{H_i}(0,t)(s)\,\mathrm{d}\hat{B}_i(s),i=1,2,\cdots,n$$

则$\hat{B}_{H_1}(t),\hat{B}_{H_2}(t),\cdots,\hat{B}_{H_n}(t)$是关于 $\hat{P}$ 的分数布朗运动,且有 $\mathrm{d}S_i(t)=(r-q_i)S_i(t)\mathrm{d}t+S_i(t)\sum_{j=1}^{n}\sigma_{ij}\mathrm{d}\hat{B}_{H_j}(t)$,$0\leqslant t\leqslant T,i=1,2,\cdots,n$。

再令

$$\hat{B}_{M_i}(t)=\sigma_{i1}\hat{B}_{H_1}(t)+\sigma_{i2}\hat{B}_{H_2}(t)+\cdots+\sigma_{in}\hat{B}_{H_n}(t),i=1,2,\cdots,n$$

则$(\hat{B}_{M_1}(t),\hat{B}_{M_2}(t),\cdots,\hat{B}_{M_n}(t))$,在 $\hat{P}$ 下服从 n 维正态分布,并且在 $\hat{P}$ 下

$$\mathrm{d}S_i(t)=(r-q_i)S_i\mathrm{d}t+S_i(t)\,\mathrm{d}\hat{B}_{M_i}(t),i=1,2,\cdots,n$$

利用 Wick 积计算得其解为

$$S_i(t)=S_i(0)\exp\left[(r-q_i)t+\hat{B}_{M_i}(t)-\frac{1}{2}\sigma_{M_i}(t)^2\right]$$
$$i=1,2,\cdots,n$$

其中

$$\sigma_{M_i}(t)^2=\sum_{l,m=1}^{n}\sigma_{il}\sigma_{im}\times\frac{\sin\left[\frac{\pi}{2}(H_l+H_m)\right]\Gamma(H_l+H_m+1)}{[\Gamma(2H_l+1)\sin(\pi H_l)\Gamma(2H_m+1)\sin(\pi H_m)]^{\frac{1}{2}}}t^{H_l+H_m}。$$

引理 1　设函数 f 满足 $E[f(B_H(T))]<\infty$,则对任意 $t\leqslant T$,有:

$$\hat{E}_t[f(\hat{B}_{M_i}(T))]=$$
$$\int_R S_i(0)\exp\left[(r-q_i)T+x_i-\frac{1}{2}\sigma_{M_i}(T)^2\right]\times g(x_i)\mathrm{d}x_i$$

其中$\hat{E}_t$ 表示概率测度 $\hat{P}$ 下的拟条件数学期望,x_i 表示$\hat{B}_{M_i}(T)$,$g(x_i)$表示为 x_i 的概率密度函数。

引理 2　任意有界 F_T 可测未定权益 $g(S_1(T),S_2(T),\cdots,S_n(T))\in L^2(P)$,其中 $g(\mu):R_+^n\to R_+$,是连续函数,则欧式未定权益 $g(S_1(T),S_2(T),\cdots,S_n(T))$的无套利价格为

$$v=\mathrm{e}^{-rT}\hat{E}[g(S_1(T),S_2(T),\cdots,S_n(T))]$$

下面在此基础上,讨论分数布朗运动下带有红利的最值期权定价公式,以及最值期权的平价公式。

3　带有红利的最值期权定价

为了方便得出最值期权定价问题,考虑任意两个股票 $S_i,S_j(i\neq j)$的最大值期权和最小值带有红利期权定价问题,假定执行价格为 K,到期日为 T。记 $b^{(i,j)}=$

$$\frac{InS_i(0)-InS_j(0)-\frac{1}{2}(\sigma_{M_i}(T)^2-\sigma_{M_j}(T)^2)-(q_i-q_j)T}{\sqrt{\sigma_{M_i}(T)^2+\sigma_{M_j}(T)^2-2\sigma_{M_iM_j}(T)}}$$

$$\sigma_{M_i}(T)^2=\sum_{l,m=1}^{n}\sigma_{il}\sigma_{im}\times\frac{\sin\left[\frac{\pi}{2}(H_l+H_m)\right]\Gamma(H_l+H_m+1)}{[\Gamma(2H_l+1)\sin(\pi H_l)\Gamma(2H_m+1)\sin(\pi H_m)]^{\frac{1}{2}}}T$$

$$\sigma_{M_iM_j}(T)=\sum_{l,m=1}^{n}\sigma_{il}\sigma_{jm}\times\frac{\sin\left[\frac{\pi}{2}(H_l+H_m)\right]\Gamma(H_l+H_m+1)}{[\Gamma(2H_l+1)\sin(\pi H_l)\Gamma(2H_m+1)\sin(\pi H_m)]^{\frac{1}{2}}}T^{H_l+H_m}$$

$$d_1^{(i)}=\frac{InS_i(0)-InK+(r-q_i)T-\frac{1}{2}\sigma_{M_i}(T)^2}{\sigma_{M_i}(T)}$$

$$d_2^{(i)}=\frac{InS_i(0)-InK+(r-q_i)T+\frac{1}{2}\sigma_{M_i}(T)^2}{\sigma_{M_i}(T)}=$$

$$d_1^{(i)}+\sigma_{M_i}(T),\eta_i=\frac{\hat{B}_{M_i}(T)}{\sigma_{M_i}(T)}$$

$$\eta_{ij}=\frac{\hat{B}_{M_i}(T)-\hat{B}_{M_j}(T)}{\sqrt{\sigma_{M_i}(T)^2+\sigma_{M_j}(T)^2-2\sigma_{M_iM_j}(T)}}$$

$$\rho_{ij}=cov(\eta_i,\eta_j)=\frac{\sigma_{M_iM_j}(T)}{\sigma_{M_i}(T)\sigma_{M_j}(T)}$$

$$\rho_{ij,i}=cov(\eta_{ij},\eta_i)=$$
$$\frac{\sigma_{M_i}(T)^2-\sigma_{M_iM_j}(T)}{\sigma_{M_i}(T)\sqrt{\sigma_{M_i}(T)^2+\sigma_{M_j}(T)^2-2\sigma_{M_iM_j}(T)}}$$

$$d^{(i,j)}=\frac{InS_i(0)-InS_j(0)}{\sqrt{\sigma_{M_i}(T)^2+\sigma_{M_j}(T)^2-2\sigma_{M_iM_j}(T)}}+$$
$$\frac{\frac{1}{2}(\sigma_{M_i}(T)^2+\sigma_{M_j}(T)^2-2\sigma_{M_iM_j}(T))-(q_i-q_j)T}{\sqrt{\sigma_{M_i}(T)^2+\sigma_{M_j}(T)^2-2\sigma_{M_iM_j}(T)}}=$$

$$b^{(i,j)}+\rho_{ij,i}\sigma_{M_i}(T),\varphi(x)=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}$$

$N(x)=$

$$\int_{-\infty}^{x}\varphi(x)\,\mathrm{d}x,\ \varphi(x,y;\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\times$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\{x^2-2\rho xy+y^2\}\right\}$$

$$N_2(x,y;\rho)=\int_{-\infty}^{x}\int_{-\infty}^{y}\varphi(x,y;\rho)\,\mathrm{d}x\mathrm{d}y$$

定理 1 关于股票 $S_i(t)$，$S_j(t)$ 的带红利的最大值欧式看涨期权，$g(S(T))=(\max\{S_i(T),S_j(T)\}-K)^+$，则其无套利价格为

$$C_{\max}=S_i(0)\mathrm{e}^{-q_iT}N_2(d^{(i,j)},d_2^{(j)};\rho_{ij,i})+$$
$$S_{\mathrm{j}}(0)\mathrm{e}^{-q_jT}N_2(d^{(j,i)},d_2^{(j)};\rho_{ji,j})-$$
$$K\mathrm{e}^{-rT}[1-N_2(-d_1^{(i)},-d_1^{(j)};\rho_{ij})]$$

证明 有引理 2 有

$$c_{\max}=\mathrm{e}^{-rT}\hat{E}(\max\{S_i(T),S_j(T)\}-K)^+=$$
$$\mathrm{e}^{-rT}\hat{E}(\max\{S_i(T),S_j(T)\}I_{\{\max\{S_i(T),S_j(T)\}>K\}})-$$
$$K\mathrm{e}^{-rT}\hat{E}(I_{\{\max\{S_i(T),S_j(T)\}>K\}})=$$
$$\mathrm{e}^{-rT}\hat{E}(S_i(T)I_{\{S_i(T)>S_j,S_i(T)>K\}})+$$
$$\mathrm{e}^{-rT}\hat{E}(S_j(T)I_{\{S_j(T)>S_i(T),S_j(T)>K\}})-$$
$$K\mathrm{e}^{-rT}\hat{P}(\{\max\{S_i(T),S_j\}>K\})=$$
$$S_i(0)\hat{E}(\exp\{-q_iT+\hat{B}_{M_i}(T)-\frac{1}{2}\sigma_{M_i}(T)^2\}\times$$
$$I_{\{S_i(T)>S_j(T),S_i(T)>K\}})+S_j(0)\hat{E}(\exp\{-q_jT+$$
$$\hat{B}_{M_j}(T)-\frac{1}{2}\sigma_{M_j}(T)^2\}\times I_{\{S_j(T)>S_i(T),S_j(T)>K\}})-$$
$$K\mathrm{e}^{-rT}[1-\hat{P}\{S_i(T)<K,S_j(T)<K\}]=$$
$$S_i(0)\hat{E}(\exp\{-q_iT+\hat{B}_{M_i}(T)-\frac{1}{2}\sigma_{M_i}(T)^2\}\times$$
$$I_{\{-\eta_{\mathrm{ij}}<b^{(i,j)},-\eta_i<d_1^{(i)}\}})+S_j(0)\hat{E}(\exp\{-q_jT+$$
$$\hat{B}_{M_j}(T)-\frac{1}{2}\sigma_{M_j}(T)^2\}\times I_{\{-\eta_{ji}<b^{(j,i)},-\eta_i<d_1^{(j)}\}})-$$
$$K\mathrm{e}^{-rT}[1-\hat{P}\{\eta_i<-d_1^{(i)},\eta_i<-d_1^{(j)}\}]=$$
$$S_i(0)\int_{-\infty}^{b^{(i,j)}}\int_{-\infty}^{d_1^{(i)}}\exp\left\{-q_iT-\sigma_{M_i}(T)y-\frac{1}{2}\sigma_{M_i}(T)^2\right\}\times$$
$$\varphi(x,y;\rho_{ji,i})\mathrm{d}y\mathrm{d}x+S_{\mathrm{j}}(0)\int_{-\infty}^{b^{(j,i)}}\int_{-\infty}^{\mathrm{d}_1^{(j)}}\exp\left\{-q_jT-\right.$$
$$\left.\sigma_{M_j}(T)y-\frac{1}{2}\sigma_{M_j}(T)^2\right\}\times\varphi(x,y;\rho_{ji,j})\mathrm{d}y\mathrm{d}x-$$
$$K\mathrm{e}^{-rT}[1-N_2(-\mathrm{d}_1^{(i)},-\mathrm{d}_1^{(j)};\rho_{ij})]=$$
$$S_i(0)\mathrm{e}^{-q_iT}\int_{-\infty}^{b^{(j,i)}+\rho_{ij,i}\sigma_{M_i}(T)}\int_{-\infty}^{\mathrm{d}_1^{(i)}+\sigma_{M_i}(T)}\varphi(x_1,y_1;\rho_{ij,i})\mathrm{d}y_1\mathrm{d}x_1+$$
$$S_{\mathrm{j}}(0)\mathrm{e}^{-q_jT}\int_{-\infty}^{b^{(j,i)}+\rho_{ji,j}\sigma_{M_j}(T)}\int_{-\infty}^{\mathrm{d}_1^{(j)}+\sigma_{M_j}(T)}\varphi(x_1,y_1;\rho_{ji,j})\mathrm{d}y_1\mathrm{d}x_1-$$
$$K\mathrm{e}^{-rT}[1-N_2(-d_1^{(i)},-d_1^{(j)};\rho_{ij})]=$$
$$S_i(0)\mathrm{e}^{-q_iT}N_2(d^{(i,j)},d_2^{(i)};\rho_{ij,i})+$$
$$S_{\mathrm{j}}(0)\mathrm{e}^{-q_jT}N_2(d^{(j,i)},d_2^{(j)};\rho_{ji,j})-K\mathrm{e}^{-rT}[1-$$
$$N_2(-d_1^{(i)},-d_1^{(j)};\rho_{ij})]$$

从而定理得证。

定理 2 关于股票 $S_i(t)$，$S_j(t)$ 的带红利的最大值欧式看跌期权，$g(S(T))=(K-\max\{S_i(T),S_j(T)\})^+$，则其无套利价格为

$$P_{\max}=K\mathrm{e}^{-rT}N_2(-d_1^{(i)},-d_1^{(j)};\rho_{ij})-$$
$$S_i(0)\mathrm{e}^{-q_iT}N_2(d^{(i,j)},-d_2^{(j)};-\rho_{ij,i})-$$
$$S_{\mathrm{j}}(0)\mathrm{e}^{-q_jT}N_2(d^{(j,i)},-d_2^{(j)};-\rho_{ji,j})$$

证明 由引理 2 有

$$p_{\max}=\mathrm{e}^{-rT}\hat{E}(K-\max\{S_i(T),S_j(T)\})^+=$$
$$K\mathrm{e}^{-rT}\hat{E}(I_{\{\max\{S_i(T),S_j(T)\}<K\}})-$$
$$\mathrm{e}^{-rT}\hat{E}(\max\{S_i(T),S_j(T)\}I_{\{\max\{S_i(T),S_j(T)\}<K\}})=$$
$$K\mathrm{e}^{-rT}\hat{P}(\{\max\{S_i(T),S_j\}<K\})-$$
$$\mathrm{e}^{-rT}\hat{E}(S_i(T)I_{\{S_i(T)>S_j,S_i(T)<K\}})-$$
$$\mathrm{e}^{-rT}\hat{E}(S_j(T)I_{\{S_j(T)>S_i(T),S_j(T)<K\}})=$$
$$K\mathrm{e}^{-rT}\hat{P}\{S_i(T)<K,S_j(T)<K\}-$$
$$S_i(0)\hat{E}(\exp\{-q_iT+\hat{B}_{M_i}(T)-\frac{1}{2}\sigma_{M_i}(T)^2\}\times$$
$$I_{\{S_i(T)>S_j(T),S_i(T)<K\}})-S_j(0)\hat{E}\left(\exp\left\{-q_jT+\right.\right.$$
$$\left.\left.\hat{B}_{M_j}(T)-\frac{1}{2}\sigma_{M_j}(T)^2\right\}\times I_{\{S_j(T)>S_i(T),S_j(T)<K\}}\right)=$$
$$K\mathrm{e}^{-rT}\hat{P}\{\eta_i<-d_1^{(i)},\eta_i<-d_1^{(j)}\}-S_i(0)\hat{E}\left(\exp\left\{-q_iT+\right.\right.$$
$$\left.\left.\hat{B}_{M_i}(T)-\frac{1}{2}\sigma_{M_i}(T)^2\right\}\times I_{\{-\eta_{ij}<b^{(i,j)},\eta_i<-d_1^{(i)}\}}\right)-$$
$$S_j(0)\hat{E}\left(\exp\left\{-q_jT+\hat{B}_{M_j}(T)-\frac{1}{2}\sigma_{M_j}(T)^2\right\}\times\right.$$
$$\left.I_{\{-\eta_{ji}<b^{(i,j)},\eta_i<-d_1^{(j)}\}}\right)=K\mathrm{e}^{-rT}N_2(-d_1^{(i)},-d_1^{(j)};\rho_{ij})-$$

$$S_i(0)\int_{-\infty}^{b(j,i)}\int_{-\infty}^{-d_1^{(i)}}\exp\left\{-q_iT+\sigma_{M_i}(T)y-\frac{1}{2}\sigma_{M_i}(T)^2\right\}\times$$
$$\varphi(x,y;-\rho_{ij,i})\mathrm{d}y\mathrm{d}x-S_j(0)\int_{-\infty}^{b(j,i)}\int_{-\infty}^{-d_1^{(j)}}\exp\left\{-q_jT+\right.$$
$$\left.\sigma_{M_j}(T)y-\frac{1}{2}\sigma_{M_j}(T)^2\right\}\times\varphi(x,y;-\rho_{ji,j})\mathrm{d}y\mathrm{d}x=$$
$$Ke^{-rT}N_2(-d_1^{(i)},-d_1^{(j)};\rho_{ij})-$$
$$S_i(0)e^{-q_iT}\int_{-\infty}^{b(i,j)+\rho_{ij,i}\sigma_{M_i}(T)}\int_{-\infty}^{-d_1^{(i)}-\sigma_{M_i}(T)}\varphi(x_1,y_1;-$$
$$\rho_{ij,i})\mathrm{d}y_1\mathrm{d}x_1-S_j(0)e^{-q_jT}\int_{-\infty}^{b(j,i)+\rho_{ji,j}\sigma_{M_j}(T)}$$
$$\int_{-\infty}^{-d_1^{(i)}-\sigma_{M_j}(T)}\varphi(x_1,y_1;-\rho_{ji,j})\mathrm{d}y_1\mathrm{d}x_1=$$
$$Ke^{-rT}N_2(-d_1^{(i)},-d_1^{(j)};\rho_{ij})-$$
$$S_i(0)e^{-q_iT}N_2(\mathrm{d}^{(i,j)},-_2^{(i)};-\rho_{ij,i})-$$
$$S_j(0)e^{-q_jT}N_2(d^{(j,i)},-d_2^{(j)};-\rho_{ji,j})$$

从而定理得证。

推论 1　关于股票 $S_i(t)$,$S_j(t)$ 的带红利的最大值欧式看涨期权与看跌期权的平价关系为

$$c_{\max}-p_{\max}=S_i(0)e^{-q_iT}N(d^{(i,j)})+S_j(0)e^{-q_jT}N(d^{(i,j)})-Ke^{-rT}$$

证明　由定理 1 与定理 2 以及正态分布性质可得。

定理 3　关于股票 $S_i(t)$,$S_j(t)$ 的带红利的最小值欧式看涨期,

$$g(S(T))=(\min\{S_i(T),S_j(T)\}-K)^+$$

则其无套利价格为

$$C_{\min}=S_i(0)e^{-q_iT}N_2(-d^{(i,j)},d_2^{(j)};-\rho_{ij,i})+$$
$$S_j(0)e^{-q_jT}N_2(-d^{(j,i)},d_2^{(j)};-\rho_{ji,j})-$$
$$Ke^{-rT}N_2(d_1^{(i)},d_1^{(j)};\rho_{ij})$$

证明　由引理 2 有

$$c_{\min}=e^{-rT}\hat{E}(\min\{S_i(T),S_j(T)\}-K)^+=$$
$$e^{-rT}\hat{E}(\min\{S_i(T),S_j(T)\}I_{\{\min\{S_i(T),S_j(T)\}>K\}})-$$
$$Ke^{-rT}\hat{E}(I_{\{\min\{S_i(T),S_j(T)\}>K\}})=$$
$$e^{-rT}\hat{E}(S_i(T)I_{\{S_j(T)>S_i,S_i(T)>K\}})+$$
$$e^{-rT}\hat{E}(S_j(T)I_{\{S_i(T)>S_j(T),S_j(T)>K\}})-$$
$$Ke^{-rT}\hat{P}(\{\min\{S_i(T),S_j\}>K\})=$$
$$S_i(0)\hat{E}\left(\exp\left\{-q_iT+\hat{B}_{M_i}(T)-\frac{1}{2}\sigma_{M_i}(T)^2\right\}\times\right.$$
$$\left.I_{\{S_i(T)<S_j(T),S_i(T)>K\}}\right)+$$
$$S_j(0)\hat{E}\left(\exp\left\{-q_jT+\hat{B}_{M_j}(T)-\frac{1}{2}\sigma_{M_j}(T)^2\right\}\times\right.$$
$$\left.I_{\{S_j(T)<S_i(T),S_j(T)>K\}}\right)-$$
$$Ke^{-rT}\hat{P}\{S_i(T)>K,S_j(T)>K\}=$$
$$S_i(0)\hat{E}\left(\exp\left\{-q_iT+\hat{B}_{M_i}(T)-\frac{1}{2}\sigma_{M_i}(T)^2\right\}\times\right.$$
$$\left.I_{\{\eta_{ij}<-b(i,j),-\eta_i<d_1^{(i)}\}}\right)+$$
$$S_j(0)\hat{E}\left(\exp\left\{-q_jT+\hat{B}_{M_j}(T)-\frac{1}{2}\sigma_{M_j}(T)^2\right\}\times\right.$$
$$\left.I_{\{\eta_{ji}<-b(j,i),-\eta_i<d_1^{(j)}\}}\right)-$$
$$Ke^{-rT}\hat{P}\{-\eta_i<d_1^{(i)},-\eta_i<d_1^{(j)}\}=$$
$$S_i(0)\int_{-\infty}^{-b(i,j)}\int_{-\infty}^{\mathrm{d}_1^{(i)}}\exp\left\{-q_iT-\sigma_{M_i}(T)y-\right.$$
$$\left.\frac{1}{2}\sigma_{M_i}(T)^2\right\}\times\varphi(x,y;-\rho_{ij,i})\mathrm{d}y\mathrm{d}x+$$
$$S_j(0)\int_{-\infty}^{-b(j,i)}\int_{-\infty}^{\mathrm{d}_1^{(j)}}\exp\{-q_jT-\sigma_{M_j}(T)y-$$
$$\left.\frac{1}{2}\sigma_{M_j}(T)^2\right\}\times\varphi(x,y;-\rho_{ji,j})\mathrm{d}y\mathrm{d}x-$$
$$Ke^{-rT}N_2(d_1^{(i)},d_1^{(j)};\rho_{ij})=S_i(0)e^{-q_iT}\times$$
$$\int_{-\infty}^{-b(i,j)-\rho_{ij,i}\sigma_{M_i}(T)}\int_{-\infty}^{d(i)_1+\sigma_{M_i}(T)}\varphi(x_1,y_1;-$$
$$\rho_{ij,i})\mathrm{d}y_1\mathrm{d}x_1+S_j(0)e^{-q_jT}\times$$
$$\int_{-\infty}^{-b(j,i)-\rho_{ji,j}\sigma_{M_j}(T)}\int_{-\infty}^{d_1^{(j)}+\sigma_{M_j}(T)}\varphi(x_1,y_1;-$$
$$\rho_{ji,j})dy_1dx_1-Ke^{-rT}N_2(d_1^{(i)},d_1^{(j)};\rho_{ij})=$$
$$S_i(0)e^{-q_iT}N_2(-d^{(i,j)},d_2^{(i)};-\rho_{ij,i})+$$
$$S_j(0)e^{-q_jT}N_2(-d^{(j,i)},d_2^{(j)};-\rho_{ji,j})-$$
$$Ke^{-rT}N_2(d_1^{(i)},d_1^{(j)};\rho_{ij})$$

从而定理得证。

定理 4　关于股票 $S_i(t)$,$S_j(t)$ 的带红利的最小值欧式看跌期权,$g(S(T))=(K-\max\{S_i(T),S_j(T)\})^+$,则其无套利价格为

$$P_{\min}=Ke^{-rT}[1-N_2(d_1^{(i)},d_1^{(j)};\rho_{ij})]-$$
$$S_i(0)e^{-q_iT}N_2(-d^{(i,j)},-d_2^{(j)};\rho_{ij,i})-$$
$$S_j(0)e^{-q_jT}N_2(-d^{(j,i)},-d_2^{(j)};\rho_{ji,j})$$

证明　由引理 2 有

$$p_{\min}=\mathrm{e}^{-rT}\hat{E}\left(K-\min\{S_i(T),S_j(T)\}\right)^{+}=$$
$$K\mathrm{e}^{-rT}\hat{E}(I_{\{\min\{S_i(T),S_j(T)\}<K\}})-$$
$$\mathrm{e}^{-rT}\hat{E}(\min\{S_i(T),S_j(T)\}I_{\{\min\{S_i(T),S_j(T)\}<K\}})=$$
$$K\mathrm{e}^{-rT}\hat{P}(\{\min\{S_i(T),S_j\}<K\})-$$
$$\mathrm{e}^{-rT}\hat{E}(S_i(T)I_{\{S_i(T)<S_j,S_i(T)<K\}})-$$
$$\mathrm{e}^{-rT}\hat{E}(S_j(T)I_{\{S_j(T)<S_i(T),S_j(T)<K\}})=$$
$$K\mathrm{e}^{-rT}[1-\hat{P}\{S_i(T)>K,S_j(T)<K\}]-$$
$$S_i(0)\hat{E}\left(\exp\left\{-q_iT+\hat{B}_{M_i}(T)-\frac{1}{2}\sigma_{M_i}(T)^2\right\}\times\right.$$
$$\left.I_{\{S_i(T)<S_j(T),S_i(T)<K\}}\right)-S_j(0)\hat{E}\left(\exp\left\{-q_jT+\right.\right.$$
$$\left.\left.\hat{B}_{M_j}(T)-\frac{1}{2}\sigma_{M_j}(T)^2\right\}\times I_{\{S_j(T)<S_i(T),S_j(T)<K\}}\right)=$$
$$K\mathrm{e}^{-rT}[1-\hat{P}\{-\eta_i<d_1^{(i)},-\eta_i<d_1^{(j)}\}]-$$
$$S_i(0)\hat{E}\left(\exp\left\{-q_iT+\hat{B}_{M_i}(T)-\frac{1}{2}\sigma_{M_i}(T)^2\right\}\times\right.$$
$$\left.I_{\{\eta_{ij}<-b^{(i,j)},\eta_i<-d_1^{(i)}\}}\right)-S_j(0)\hat{E}\left(\exp\left\{-q_jT+\right.\right.$$
$$\left.\left.\hat{B}_{M_j}(T)-\frac{1}{2}\sigma_{M_j}(T)^2\right\}I_{\{\eta_{ji}<-b^{(j,i)},\eta_i<-d_1^{(j)}\}}\right)=$$
$$K\mathrm{e}^{-rT}[1-N_2(d_1^{(i)},d_1^{(j)};\rho_{ij})]-$$
$$S_i(0)\mathrm{e}^{-q_iT}\int_{-\infty}^{-b(i,j)}\int_{-\infty}^{-\mathrm{d}_1^{(i)}}\exp\{\sigma_{M_i}(T)-$$
$$\frac{1}{2}\sigma_{M_i}(T)^2\}\mathrm{d}y\mathrm{d}x-S_{\mathrm{j}}(0)\mathrm{e}^{-q_jT}$$
$$\int_{-\infty}^{-b(j,i)-\rho_{ji,j}\sigma_{M_j}(T)}\int_{-\infty}^{-d_1^{(j)}-\sigma_{M_j}(T)}\exp\left\{\sigma_{M_j}(T)-\right.$$
$$\left.\frac{1}{2}\sigma_{M_j}(T)^2\right\}\mathrm{d}y\mathrm{d}x=K\mathrm{e}^{-rT}[1-$$
$$N_2(d_1^{(i)},d_1^{(j)};\rho_{ij})]-S_i(0)\times$$
$$\mathrm{e}^{-q_iT}\int_{-\infty}^{-b(i,j)-\rho_{ij,i}\sigma_{M_i}(T)}\int_{-\infty}^{-d_1^{(i)}-\sigma_{M_i}(T)}\varphi(x_1,y_1;\rho_{ij,i})\mathrm{d}y_1\mathrm{d}x_1-$$
$$S_{\mathrm{j}}(0)\mathrm{e}^{-q_jT}\int_{-\infty}^{-b(j,i)-\rho_{ji,j}\sigma_{M_j}(T)}\int_{-\infty}^{-d_1^{(j)}-\sigma_{M_j}(T)}$$
$$\varphi(x_1,y_1;\rho_{ji,j})\mathrm{d}y_1\mathrm{d}x_1=$$
$$K\mathrm{e}^{-rT}[1-N_2(d_1^{(i)},d_1^{(j)};\rho_{ij})]-$$
$$S_i(0)\mathrm{e}^{-q_iT}N_2(-d^{(i,j)},-d_2^{(i)};\rho_{ij,i})-$$
$$S_{\mathrm{j}}(0)\mathrm{e}^{-q_jT}N_2(-d^{(j,i)},-d_2^{(j)};\rho_{ji,j})$$

从而定理得证。

推论 2 关于股票 $S_i(t)$,$S_j(t)$ 的带红利的最小值欧式看涨期权与看跌期权的平价关系为

$$c_{\min}-p_{\min}=S_i(0)\mathrm{e}^{-q_iT}N(-d^{(i,j)})+S_j(0)\mathrm{e}^{-q_jT}N(-d^{(i,j)})-K\mathrm{e}^{-rT}$$

证明由定理 3 和定理 4 以及正态分布性质可得。

推论 3 关于股票 $S_i(t)$,$S_j(t)$ 的带红利的最大值和最小值欧式期权价格关系为

$$[c_{\max}-p_{\max}]+[c_{\min}-p_{\min}]=S_i(0)\mathrm{e}^{-q_iT}+S_j(0)\mathrm{e}^{-q_jT}-2K\mathrm{e}^{-rT}$$

证明 由推论 1 和推论 2 以及正态分布性质可得。

4 结束语

最值期权是一种新型期权,为了研究带有红利的最值期权定价问题,首先考虑的是股票价格满足带有红利率由分数布朗运动驱动的随机微分方程,并且假定红利率为常数,通过利用风险中性原则和拟鞅方法,得出分数布朗运动下的最值期权的定价公式以及平价公式。

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The Minimum or Maximum Option Pricing with Dividend under Fractional Brownian Motion

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Abstract: According to the investor's purchase of a listed company's stock and the investment in this company while enjoying the company's right for dividends, based on this actual situation, this paper considers the pricing of the dividends on the sale of the most valued options. First, it is assumed that the price of the stock obeys the stochastic differential equation driven by fractional Brownian motion. The Radon-Nikodim derivative theorem and the multidimensional Girsanov's theorem are used to define the risk neutral probability measure. At the same time, the stochastic differential equation for each stock price of the multidimensional fractional Brown motion under the risk neutral probability measure is established. Based on the Wick product principle, the price formula of each stock price is obtained. Finally, the risk-neutral pricing method is used to obtain call and put option pricing formulas and parity formulas with maximum and minimum dividends under fractional Brownian motion. The obtained results can provide a theoretical reference for studying the issue of valuation of options under consideration of the actual situation of stock dividends payment.

Key words: fractional Brownian motion; payment of dividend; the minimum or maximum option; risk neutral pricing

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